7 Stability of Beams

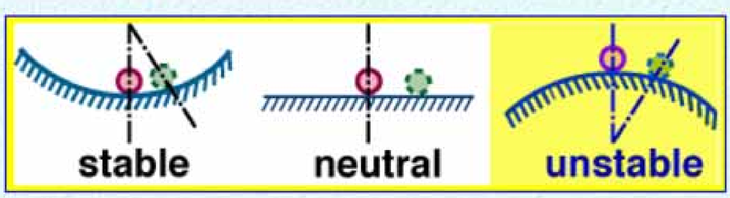
Concept of Stability

- A configuration (equilibrium state) is stable if a small perturbation (disturbance) results in a small change in the configuration.

- The original configuration is restored upon the removal of the disturbance.

Example: A small ball, displaced from its original configuration, on a surface.

Types of Equilibrium: stable, neutral, unstable.

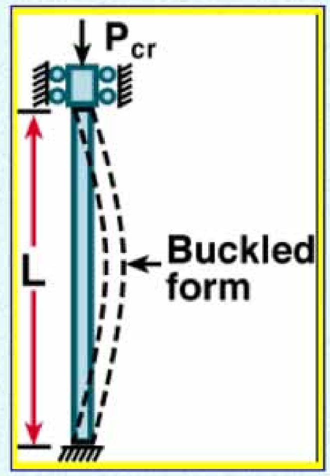


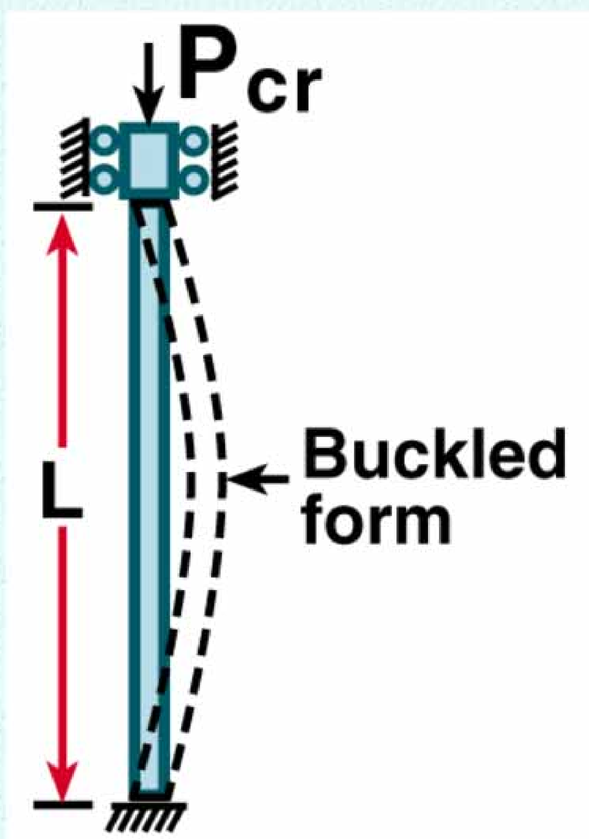
Stability Criteria

- **Static criterion:** equilibrium method - based on studying the equilibrium of an adjacent configuration.

- **Energy criterion:** energy method static - based on studying the changes in the total potential energy in going from the original to a neighboring configuration.

- **Dynamic criterion:** dynamic method - based on studying the motion resulting from a small disturbance.





Definitions

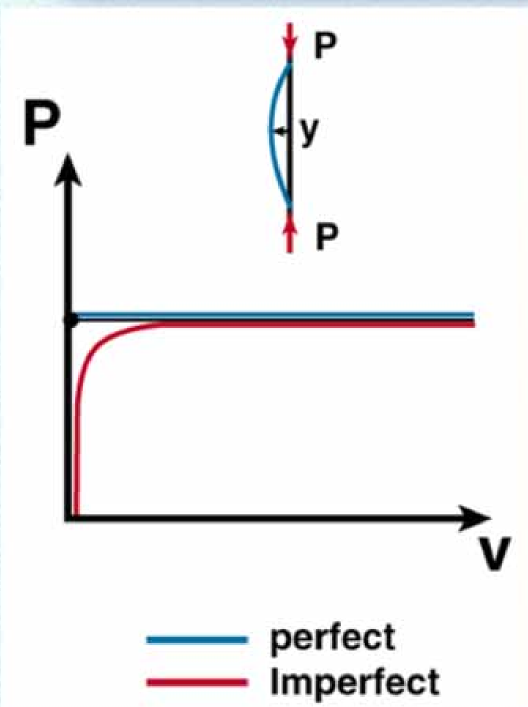
- Rigid bar - spring systems - systems with finite number of degrees of freedom

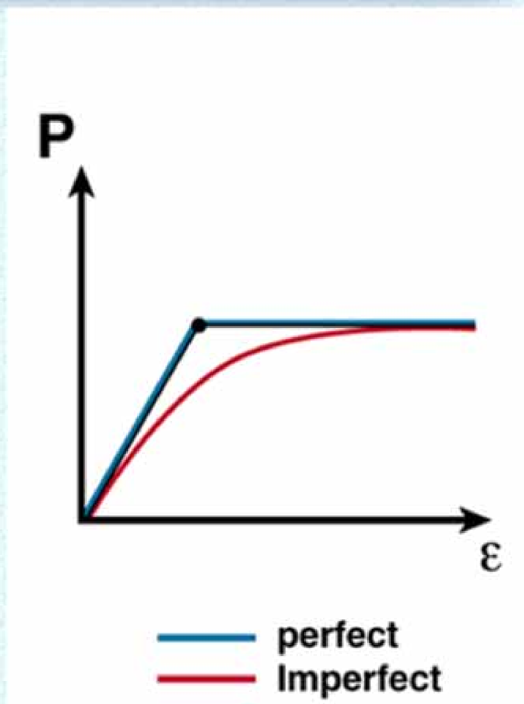
  Algebraic eigenvalue problem (system of homogeneous algebraic equations)

- Deformable columns - distributed systems

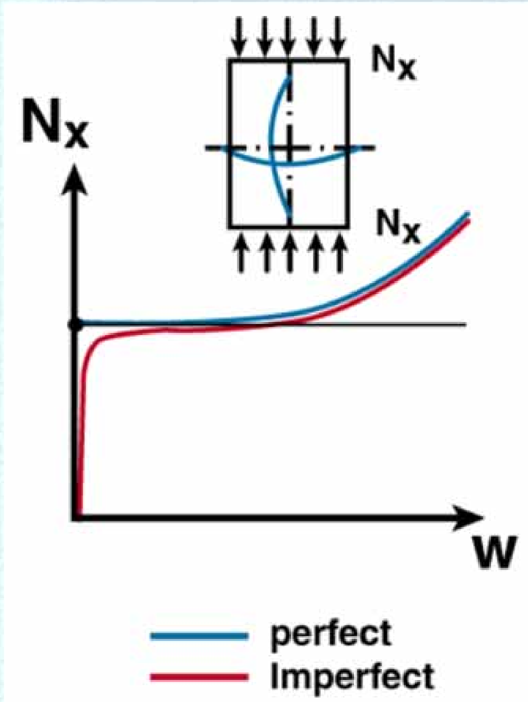
  Homogeneous differential equation and homogeneous boundary conditions

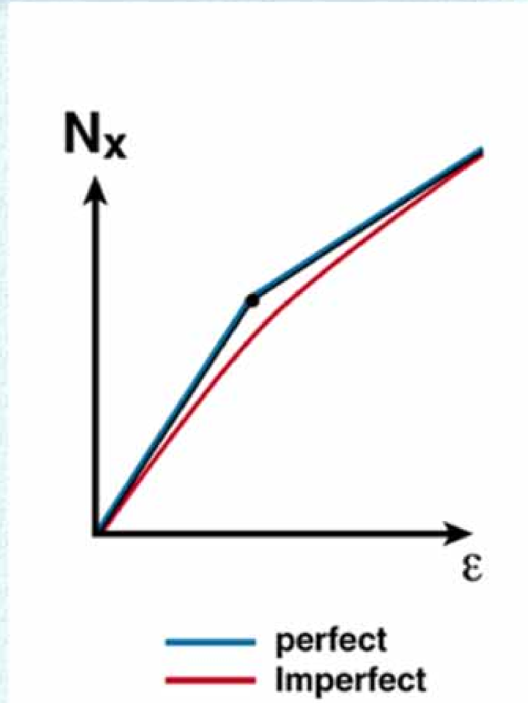
Structures with and without Postbuckling Strength



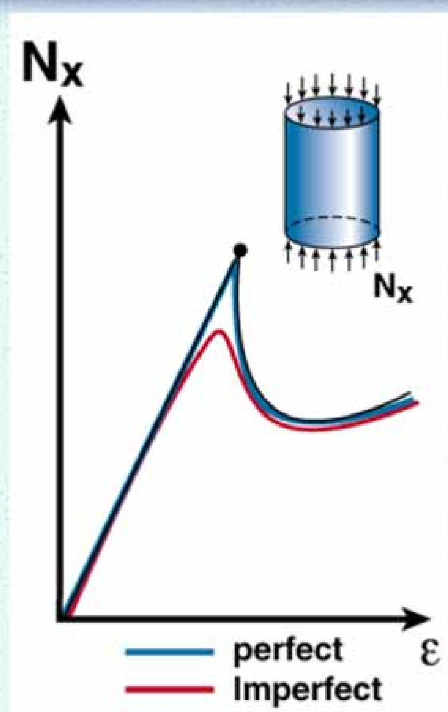


Structures with and without Postbuckling Strength





Structures with and without Postbuckling Strength

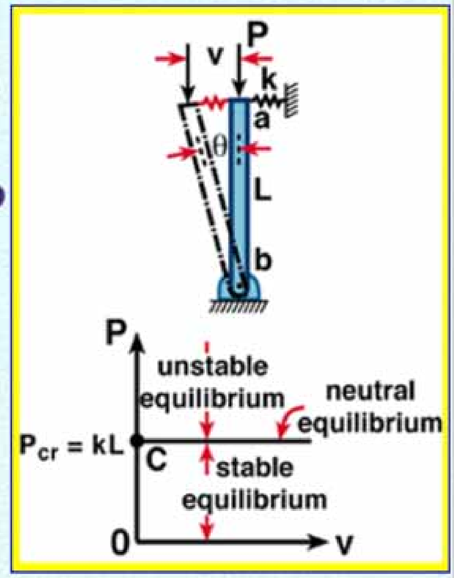


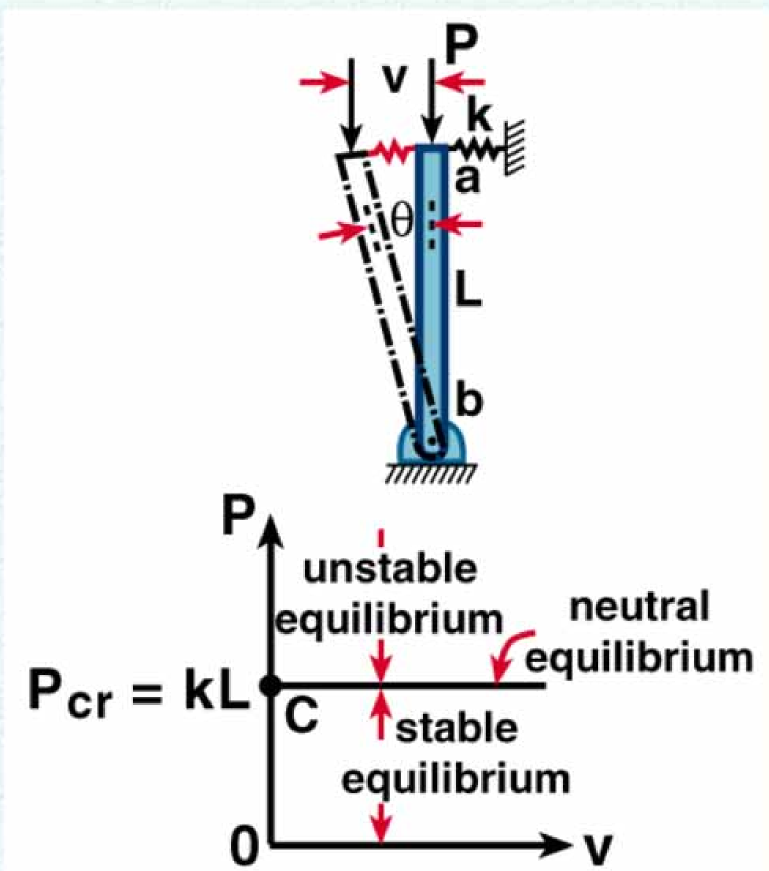
Stability of Columns (or Struts)

Axially Loaded Rigid Bar with a Pin and a Linear Elastic Support

a) Equilibrium (Euler) Method

- Studying the equilibrium of an adjacent configuration (to the original one)
  Spring force = k V
  where k = spring stiffness
- Moments about the support at b
  - Disturbing moment = Pv
  - Restoring moment = kvL



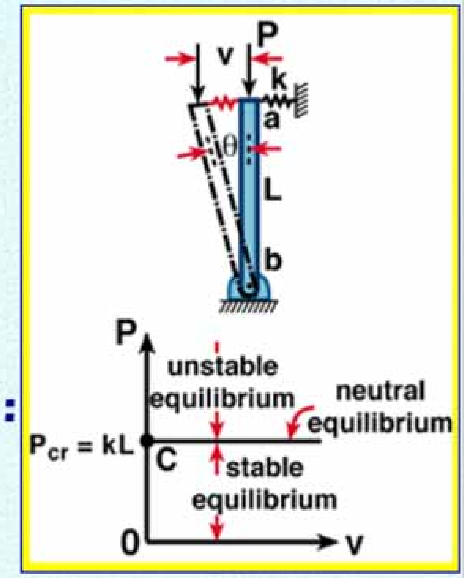


Stability of Columns (or Struts)

Axially Loaded Rigid Bar with a Pin and a Linear Elastic Support

a) Equilibrium (Euler) Method

- Moments about the support at b
  - Disturbing moment = Pv
  - Restoring moment = kvL
- Three different states of equilibrium can be identified:
  - Stable equilibrium P<kL
  - Unstable equilibrium P>kL
  - Neutral equilibrium P = kL

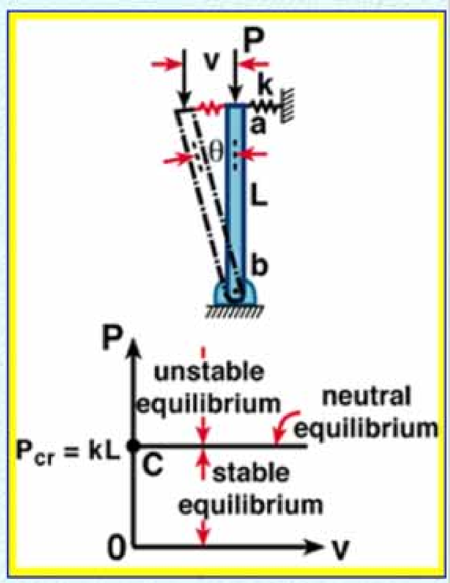


Stability of Columns (or Struts)

Axially Loaded Rigid Bar with a Pin and a Linear Elastic Support

a) Equilibrium (Euler) Method

- Neutral equilibrium corresponds to the onset of buckling - presence of two equilibrium states:
  - Original - unstable
  - Deformed - stable

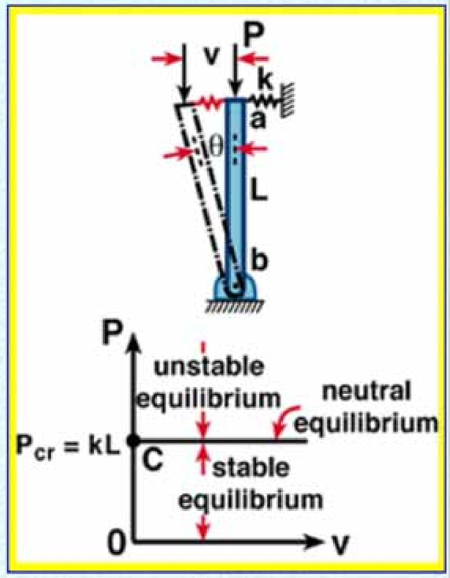


Stability of Columns (or Struts)

Axially Loaded Rigid Bar with a Pin and a Linear Elastic Support

a) Equilibrium (Euler) Method

- This is referred to as bifurication of equilibrium.
- The value of P associated with the neutral equilibrium state is referred to as bifurcation (or Euler) buckling load.



Stability of Columns (or Struts)

Axially Loaded Rigid Bar with a Pin and a Linear Elastic Support

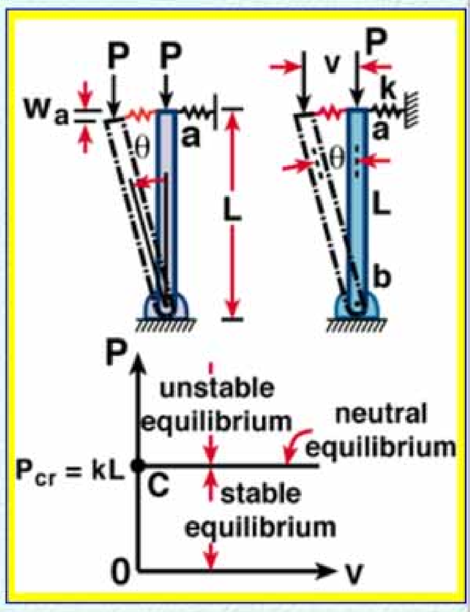
b) Energy Method

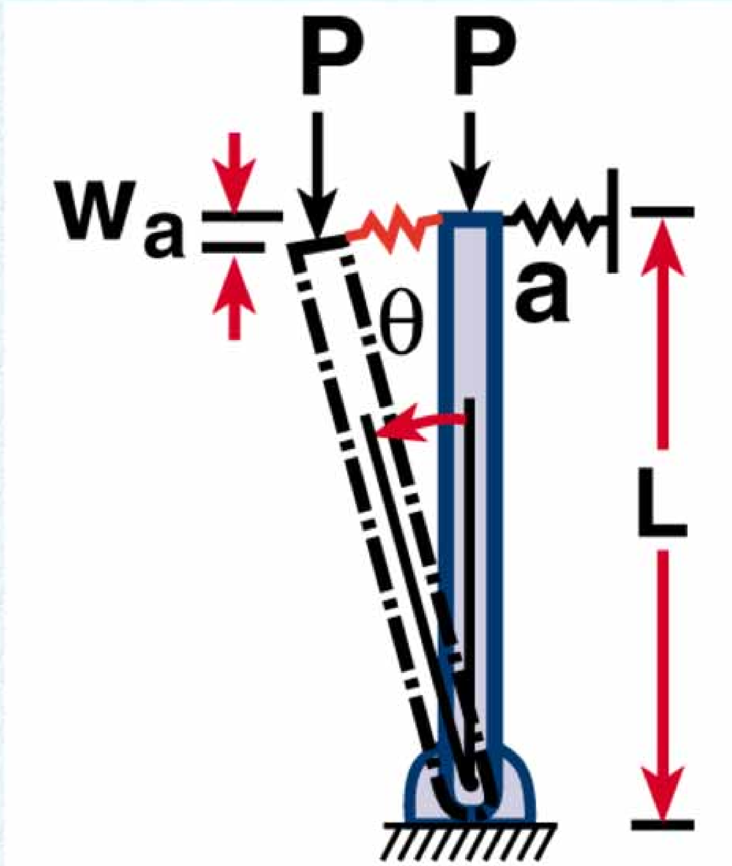
$\mathbf{w}_a$ = vertical displacement of point a = L(1-cosθ)

$$\boxed{= L\left[1-\left(1-\frac{\theta^2}{2}+\ldots\right)\right]=\frac{1}{2}L\,\theta^2}$$

$\Delta \mathbf{w}$ = work done by the axial force P = P $\mathbf{w}_a$

$$= \frac{1}{2} P\,L\,\theta^{\,2}$$





Stability of Columns (or Struts)

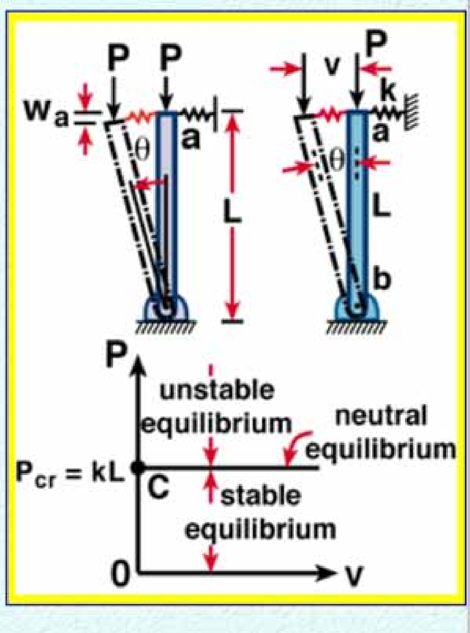
Axially Loaded Rigid Bar with a Pin and a Linear Elastic Support

b) Energy Method

ΔW = work done by the axial force P = $P\,w_a$

$= \frac{1}{2} P L \theta^2$

ΔU = strain energy in the spring $= \frac{1}{2} K (L\theta)^2$



Stability of Columns (or Struts)

Axially Loaded Rigid Bar with a Pin and a Linear Elastic Support

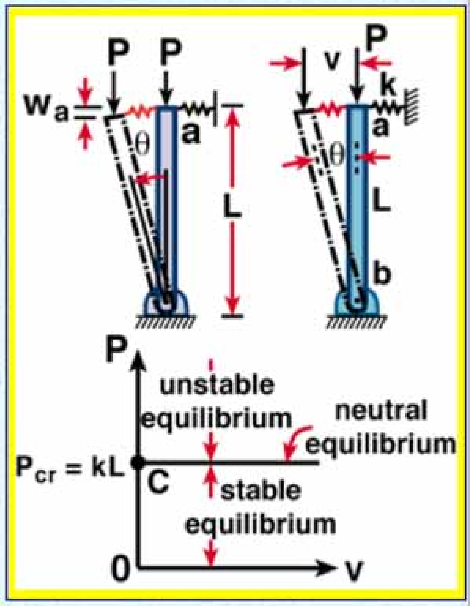
b) Energy Method

The three different equilibrium states correspond to:

$\Delta U > \Delta W$ - **stable equilibrium**

$\Delta U < \Delta W$ - **unstable equilibrium**

$\Delta U = \Delta W$ - **neutral equilibrium**



Stability of Columns (or Struts)

Axially Loaded Rigid Bar with a Pin and a Linear Elastic Support

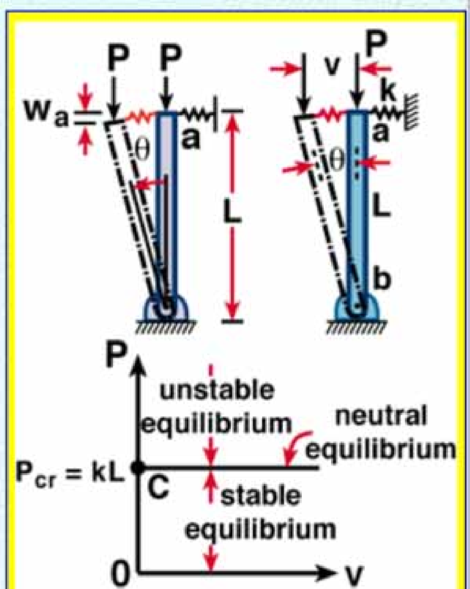
b) Energy Method

Bifurcation buckling (critical) load corresponds to:

$$\Delta U = \Delta W$$

or

$$\boxed{P_{cr} = \kappa L}$$



Stability of Columns (or Struts)

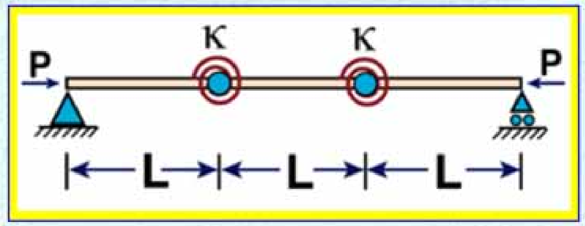
c) Equilibrium Method

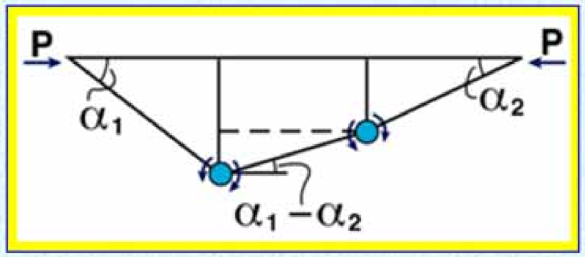
Equilibrium of deformed configuration

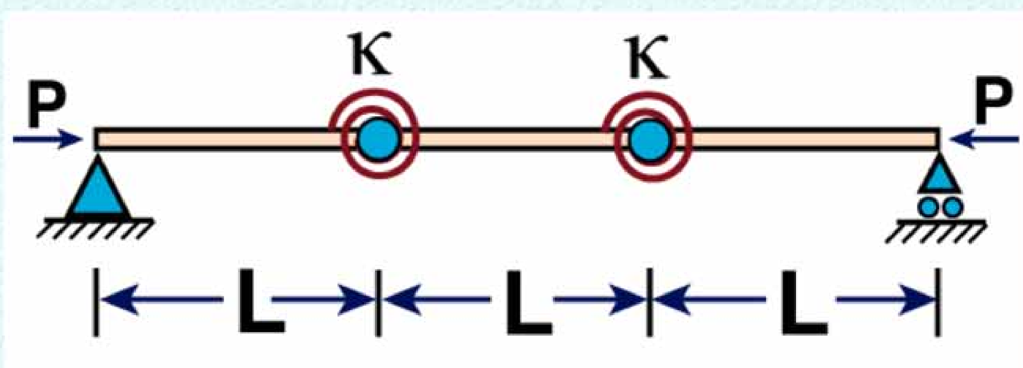
$$\boxed{P\,\alpha_2\,L-(2\alpha_2-\alpha_1)\kappa=0}$$

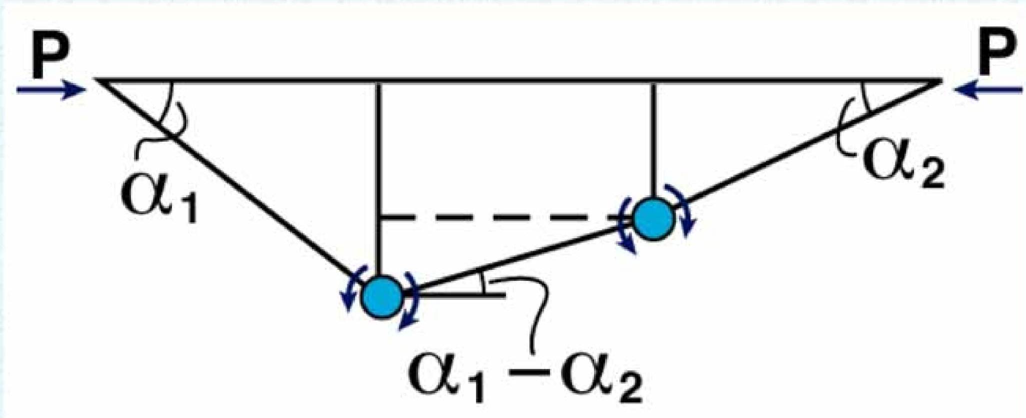
$$\boxed{P\,\alpha_1\,L-(2\alpha_1-\alpha_2)\kappa=0}$$

or

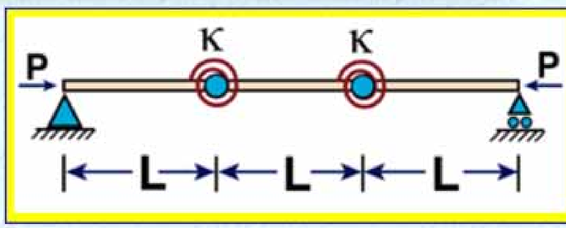








Stability of Columns (or Struts)



c) Equilibrium Method

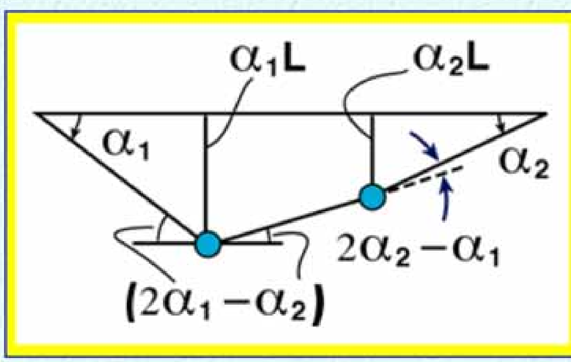
Equilibrium of deformed configuration

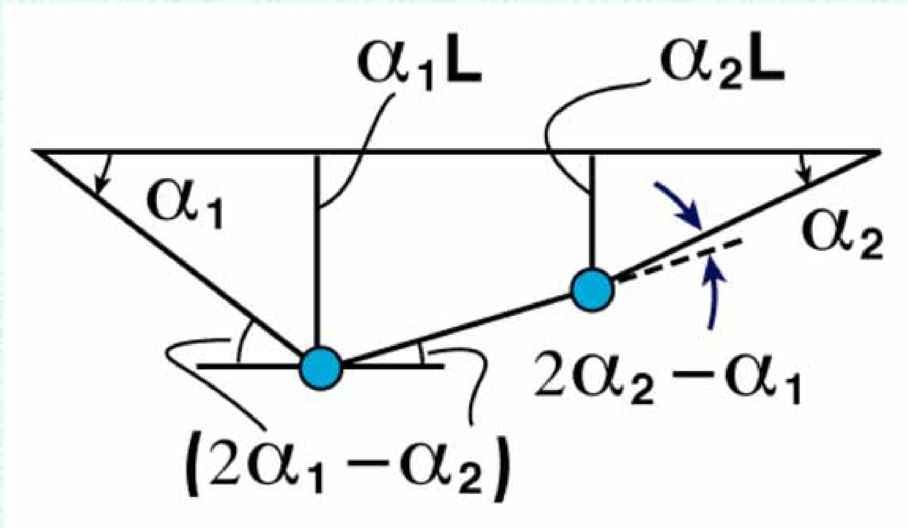
$$\boxed{P\,\alpha_2 L - (2\alpha_2 - \alpha_1)\kappa = 0}$$

$$\boxed{P\,\alpha_1 L - (2\alpha_1 - \alpha_2)\kappa = 0}$$

or

$$\begin{bmatrix} PL - 2\kappa & \kappa \\ \kappa & PL - 2\kappa \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = 0$$



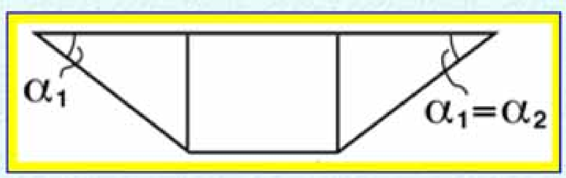


Stability of Columns (or Struts)

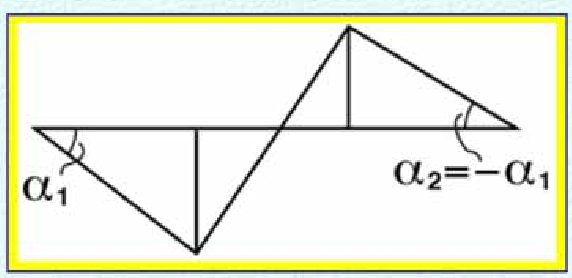
Characteristic Equation (Quadratic Equation in P)

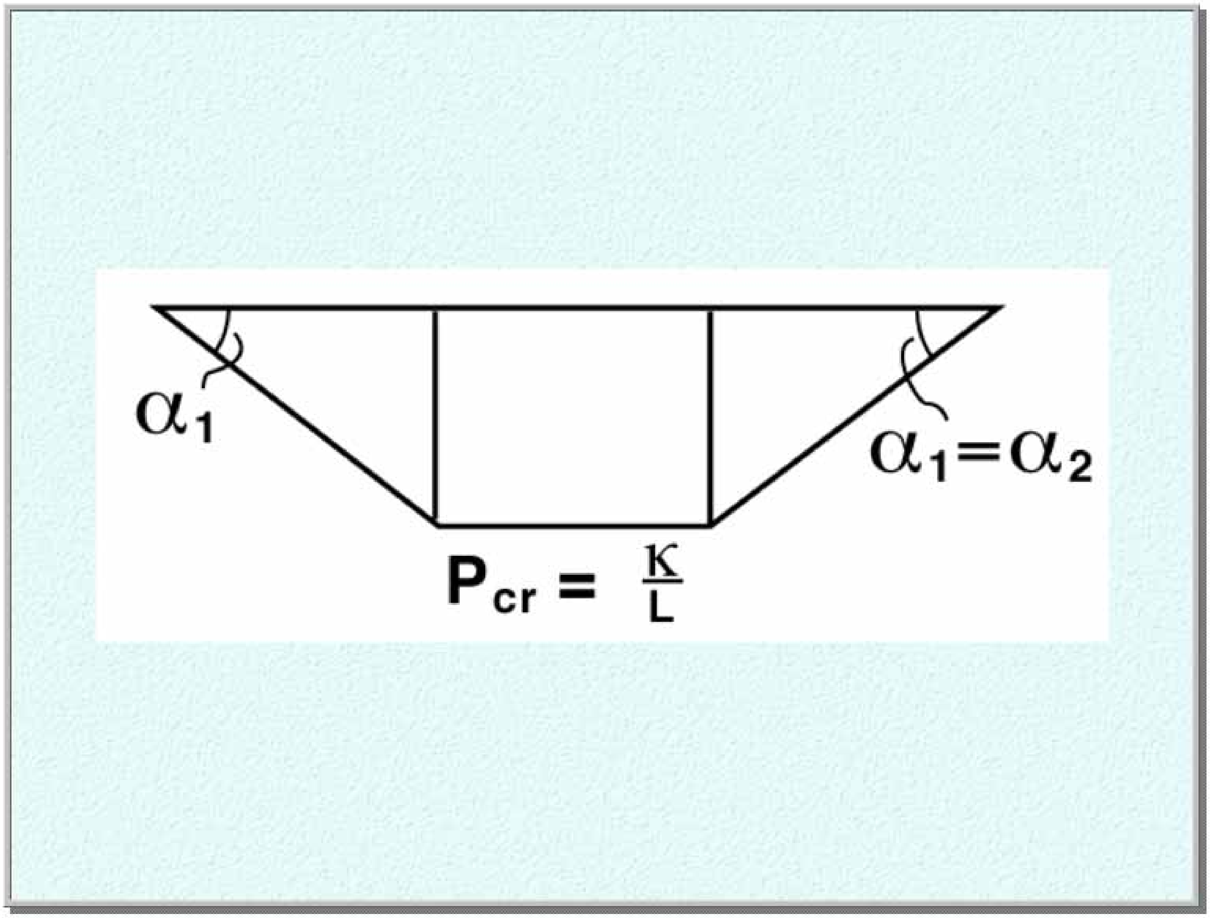
$$\boxed{L^2 P^2 - 4\,\kappa L\,P + 3\kappa^2 = 0}$$

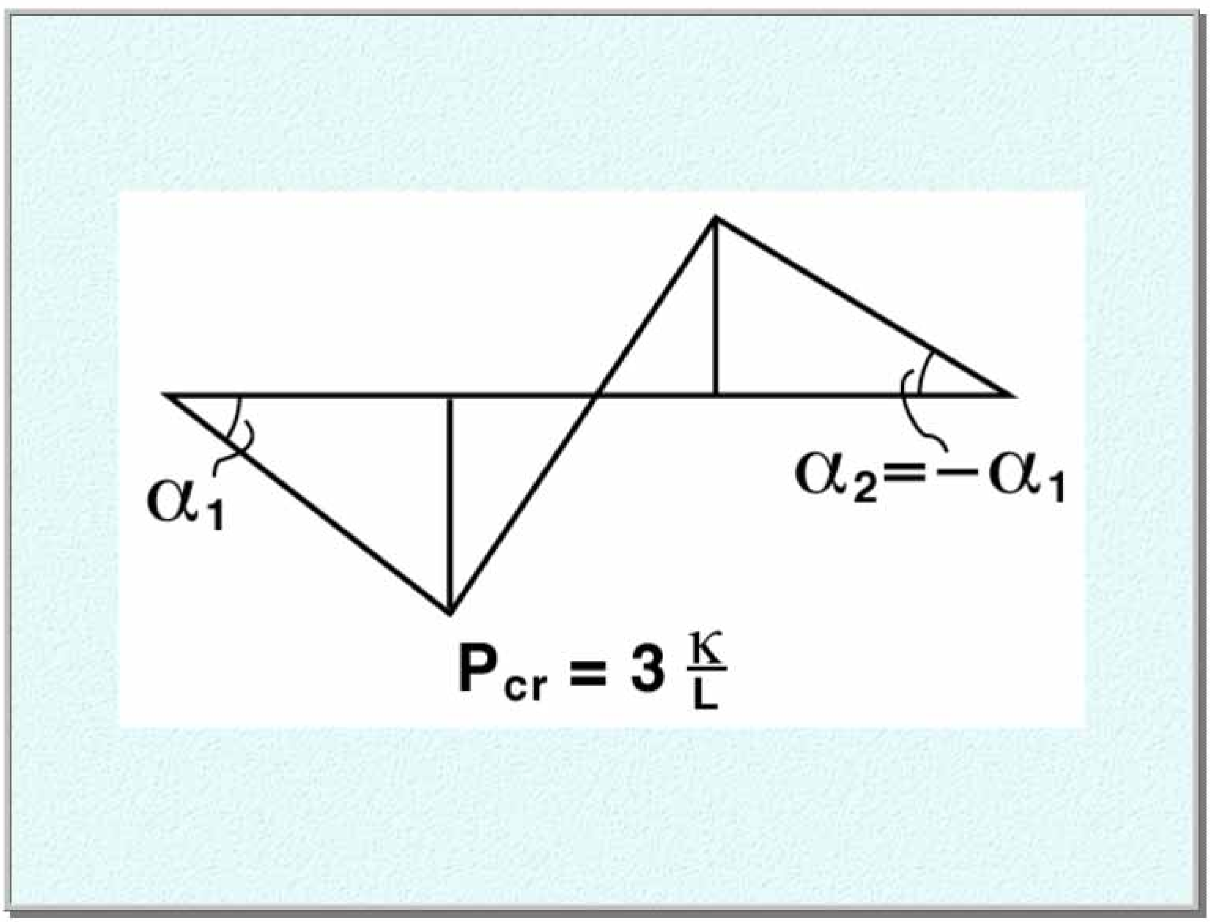
$$\boxed{P_{cr_1} = \frac{\kappa}{L} \rightarrow \alpha_1 = \alpha_2}$$



$$\boxed{P_{cr_2} = 3\,\frac{\kappa}{L} \rightarrow \alpha_1 = -\alpha_2}$$




α_1
$\alpha_1 = \alpha_2$
$P_{cr} = \frac{K}{L}$


α_1
$\alpha_2 = -\alpha_1$
$P_{cr} = 3\,\frac{K}{L}$

Stability of Columns (or Struts)

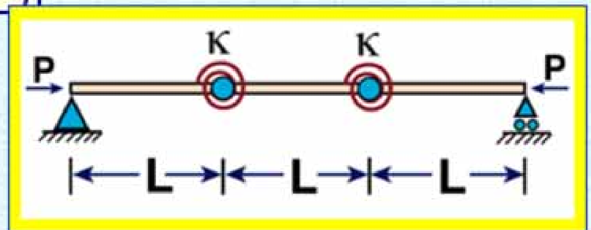
d) Energy Method

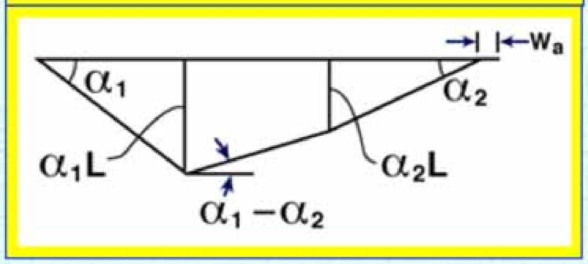
$$W_a = L\left(1 - \cos\alpha_2\right) + L\left(1 - \cos\alpha_1\right)$$

$$+ L\left[1 - \cos\left(\alpha_1 - \alpha_2\right)\right]$$

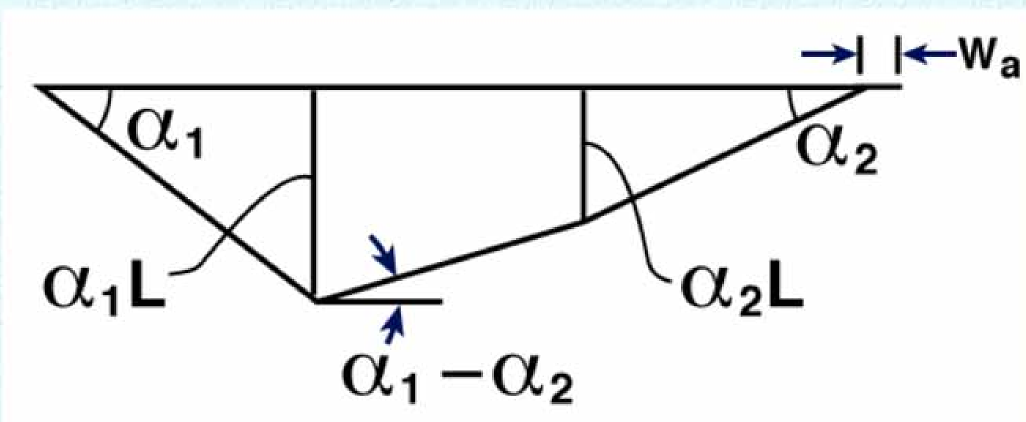
$$\cong \frac{L}{2}\left[\alpha_2^2 + \alpha_1^2 + \left(\alpha_1 - \alpha_2\right)^2\right]$$

$$= L\left(\alpha_1^2 - \alpha_1\alpha_2 + \alpha_2^2\right)$$





W = work done by axial force $= Pw_a$



Stability of Columns (or Struts)

d) Energy Method

U = *strain energy in springs*

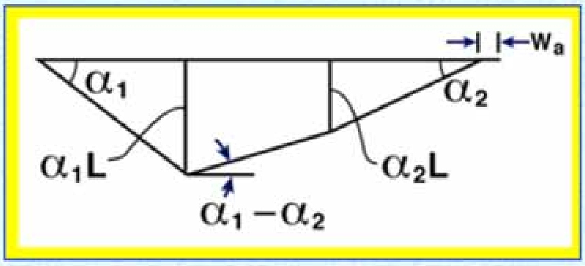
$$\boxed{= \frac{1}{2}\kappa\left(2\alpha_2 - \alpha_1\right)^2 + \frac{1}{2}\kappa\left(2\alpha_1 - \alpha_2\right)^2}$$

Π = total potential energy

$$\boxed{= U - W}$$

For stable equilibrium Π *is minimum*

$$\boxed{\frac{\partial \Pi}{\partial \alpha_1} = \frac{\partial \Pi}{\partial \alpha_2} = 0}$$



Stability of Columns (or Struts)

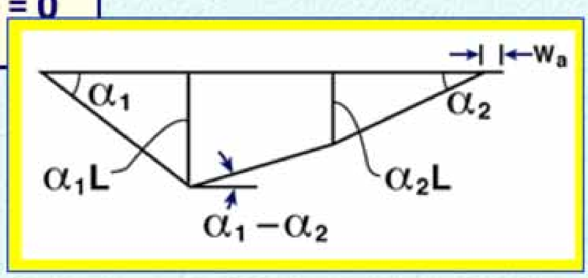
d) Energy Method

$$\boxed{\frac{\partial \Pi}{\partial \alpha_1} = 0 \rightarrow -\kappa\left(2\alpha_2 - \alpha_1\right) + 2\kappa\left(2\alpha_1 - \alpha_2\right) - PL\left(2\alpha_1 - \alpha_2\right) = 0}$$

$$\boxed{\frac{\partial \Pi}{\partial \alpha_2} = 0 \rightarrow 2\kappa\left(2\alpha_2 - \alpha_1\right) - \kappa\left(2\alpha_1 - \alpha_2\right) - PL\left(-\alpha_1 + 2\alpha_2\right) = 0}$$

or

$$\boxed{\begin{bmatrix} -2PL + 5\kappa & PL - 4\kappa \\ PL - 4\kappa & -2PL + 5\kappa \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = 0}$$



Stability of Columns (or Struts)

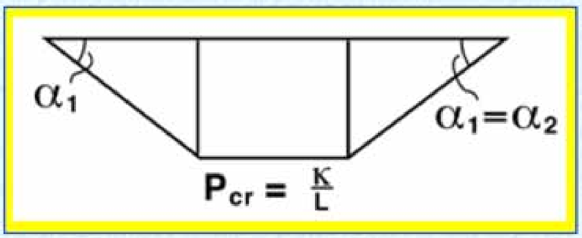
d) Energy Method

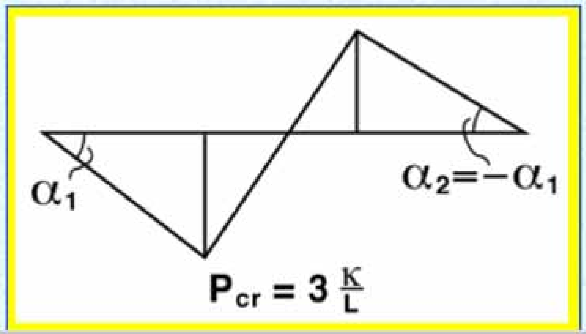
Characteristic Equation (Quadratic in P):

$$\boxed{L^2P^2 - 4L\kappa P + 3\kappa^2 = 0}$$

$$\boxed{P_{cr_1} = \frac{\kappa}{L}}$$

$$\boxed{P_{cr_2} = 3\frac{\kappa}{L}}$$





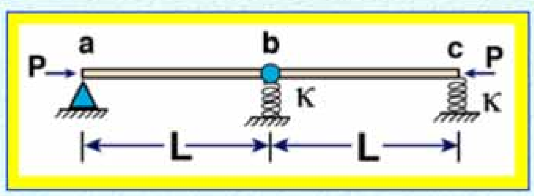
Stability of Columns (or Struts)

Two-Degrees of Freedom System

Equilibrium Method

Bending moment at a = 0

$$\boxed{P\,v_2 - \kappa v_2 \cdot 2L - \kappa v_1 \cdot L = 0}$$

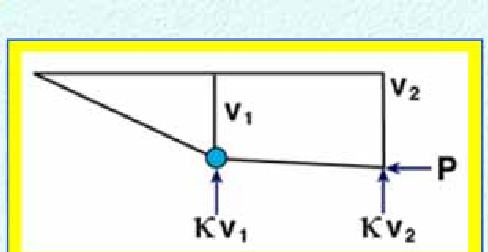


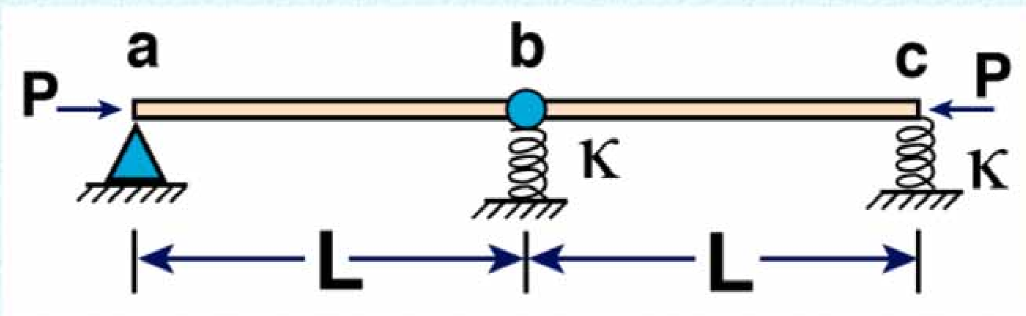
Bending moment at b = 0

$$\boxed{P(v_2 - v_1) - \kappa v_2 \cdot L = 0}$$

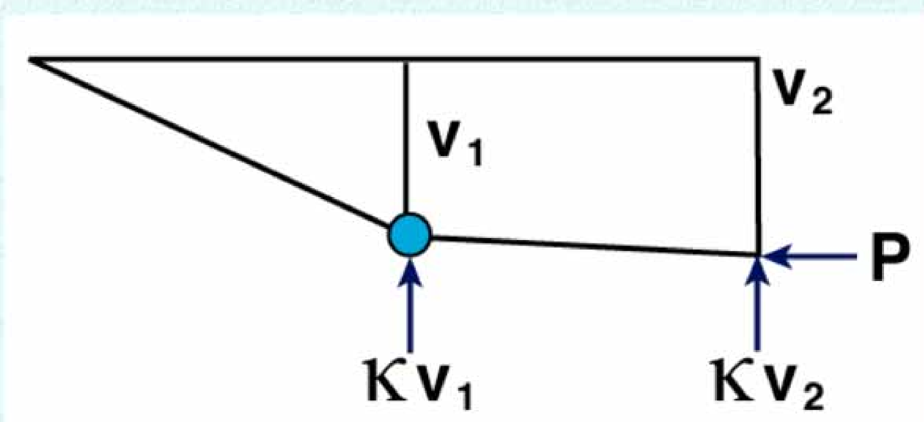
or

$$\boxed{\begin{bmatrix} -\kappa L & P - 2\kappa L \\ -P & P - \kappa L \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = 0}$$





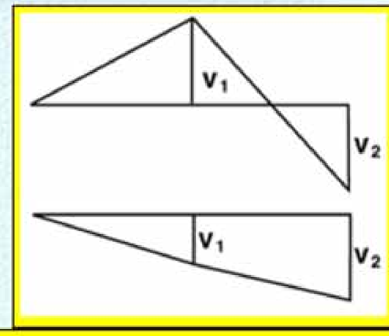
a
b
c
P
P
K
K
L
L



v_2
v_1
P
Kv_1
Kv_2

Stability of Columns (or Struts)

Two-Degrees of Freedom System

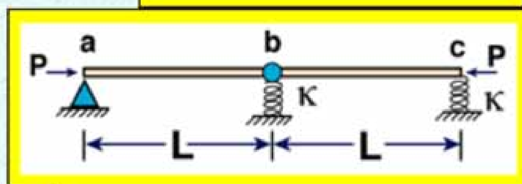


Characteristic Equation

$$\boxed{P^2 - 3\kappa LP + \kappa^2 L^2 = 0}$$

$$\boxed{P = 0.382\,\kappa L}$$

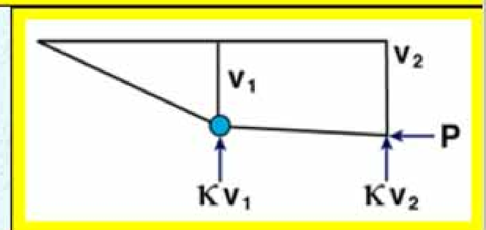
or $$\boxed{P = 2.618\,\kappa L}$$

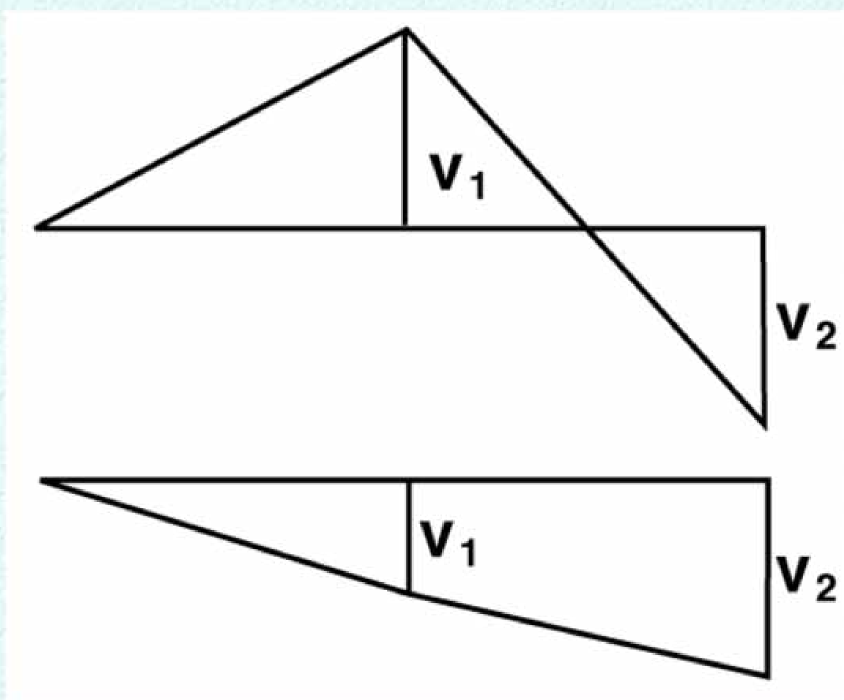


Energy Method

$$\boxed{w_c = L(1 - \cos\theta_1) + L(1 - \cos\theta_2)}$$

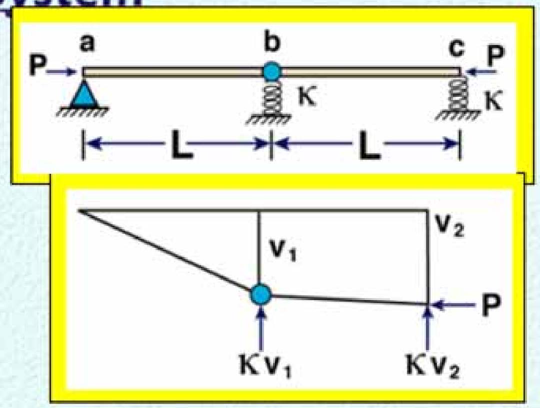
$$\boxed{= \frac{1}{2} L\,\theta_1^2 + \frac{1}{2} L\,\theta_2^2}$$





Stability of Columns (or Struts)

Two-Degrees of Freedom System



where

$$\theta_1 = \frac{v_1}{L}$$

$$\theta_2 = \frac{v_2 - v_1}{L}$$

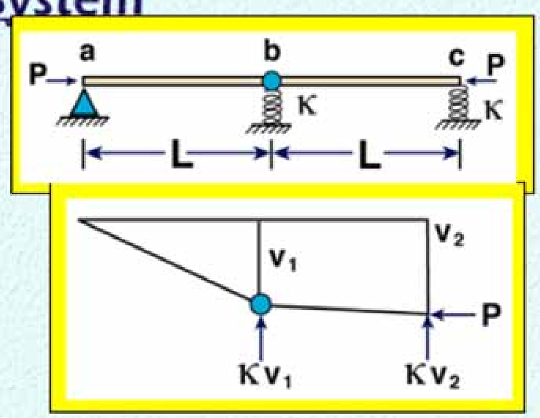
W = work done by axial force

$$= \frac{1}{2} PL\left[\left(\frac{v_1}{L}\right)^2 + \left(\frac{v_2 - v_1}{L}\right)^2\right]$$

U = strain energy in springs

Stability of Columns (or Struts)

Two-Degrees of Freedom System



$$= \frac{1}{2} \kappa v_1^2 + \frac{1}{2} \kappa v_2^2$$

$$\Pi = U - W$$

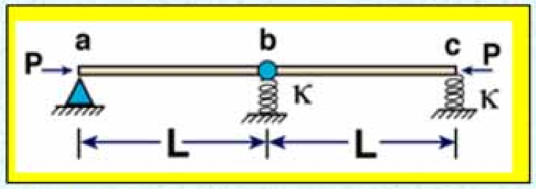
$$\frac{\partial \Pi}{\partial v_1} = \frac{\partial \Pi}{\partial v_2} = 0$$

$$\begin{bmatrix} \kappa - \frac{2P}{L} & \frac{P}{L} \\ \frac{P}{L} & \kappa - \frac{P}{L} \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = 0$$

Stability of Columns (or Struts)

Characteristic Equation

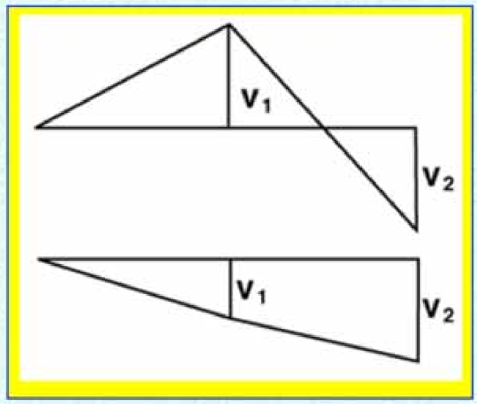
$$\boxed{\frac{P^2}{L^2} - \frac{3\kappa P}{L} + \kappa^2 = 0}$$



$$\boxed{P = 0.382\,\kappa L}$$

or

$$\boxed{P = 2.618\,\kappa L}$$

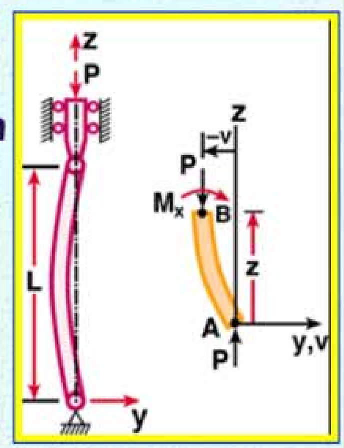


Stability of Columns (or Struts)

Axially Loaded Slender Column with Simply Supported Ends

Equilibrium Method

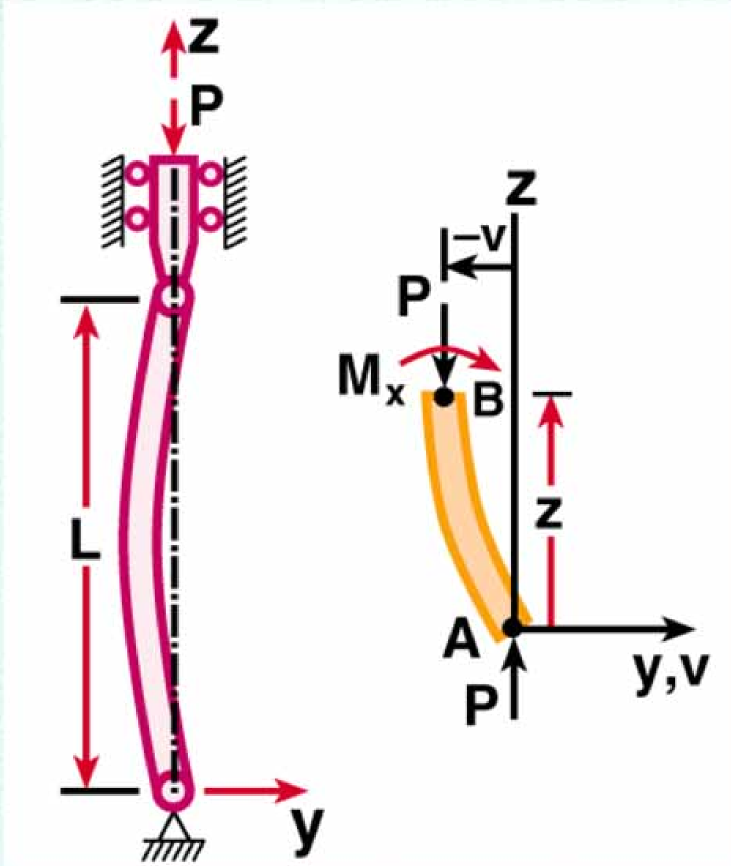
Studying the equilibrium of a portion in an adjacent configuration



$$M = -P\,v = EI\,\frac{d^2v}{dz^2}$$

or

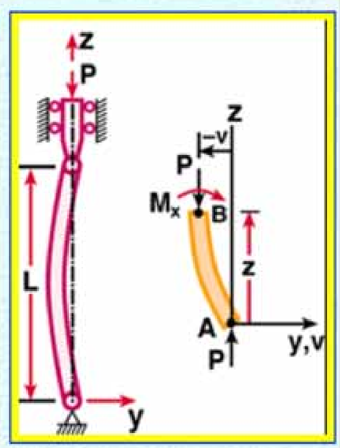
$$\boxed{\frac{d^2v}{dz^2} + \frac{P}{EI}\,v = 0}$$



Stability of Columns (or Struts)

Axially Loaded Slender Column with Simply Supported Ends

$$\boxed{\begin{aligned} M &= -P\,v \\ &= EI\,\frac{d^2v}{dz^2} \end{aligned}}$$



or

$$\boxed{\frac{d^2v}{dz^2} + \frac{P}{EI}\,v = 0}$$

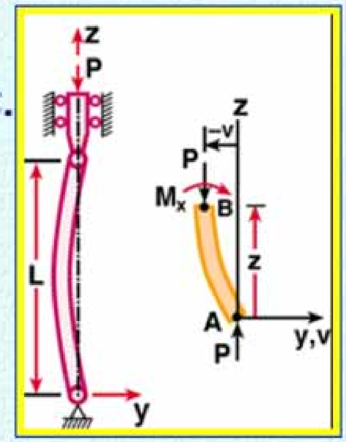
which is a homogeneous ordinary differential equation.

Stability of Columns (or Struts)

<u>Boundary Conditions</u>

At $z = 0$ and L, $v = 0$

Homogeneous boundary conditions.



General solution of the differential equation is:

$$v = A \sin \sqrt{\frac{P}{EI}}\, z + B \cos \sqrt{\frac{P}{EI}}\, z$$

where A and B are constants.

Applying the boundary conditions

At $z=0$, $v=0 \longrightarrow B=0$

At $z=L$, $v=0 \longrightarrow A \sin \sqrt{\frac{P}{EI}}\, L = 0$

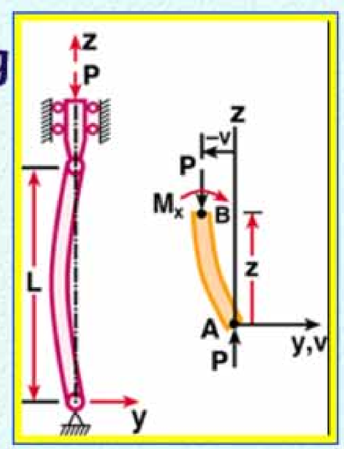
Stability of Columns (or Struts)

Applying the boundary conditions

At $z=0$, $v=0 \longrightarrow B=0$

At $z=L$, $v=0 \longrightarrow A \sin \sqrt{\frac{P}{EI}}\, L = 0$

Either $A = 0 \longrightarrow v = 0$ no buckling



or,

$$\sin \sqrt{\frac{P}{EI}}\, L = 0 \longrightarrow \sqrt{\frac{P}{EI}}\, L = n\pi$$

where n is an integer (n=1,2,3...)

$$P = \frac{n^2\pi^2 EI}{L^2}$$

Stability of Columns (or Struts)

Either $A = 0 \longrightarrow v = 0$ no buckling

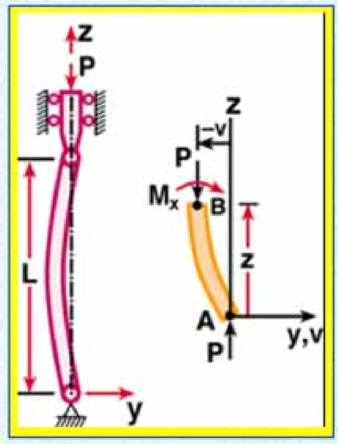
or, $\boxed{\sin\sqrt{\frac{P}{EI}}L = 0 \longrightarrow \sqrt{\frac{P}{EI}}L = n\pi}$

where n is an integer (n=1,2,3...)

$$\boxed{P = \frac{n^2\pi^2 EI}{L^2}}$$

Lowest critical load corresponds to n = 1.

$$\boxed{P_{cr} = \frac{\pi^2 EI}{L^2}}$$

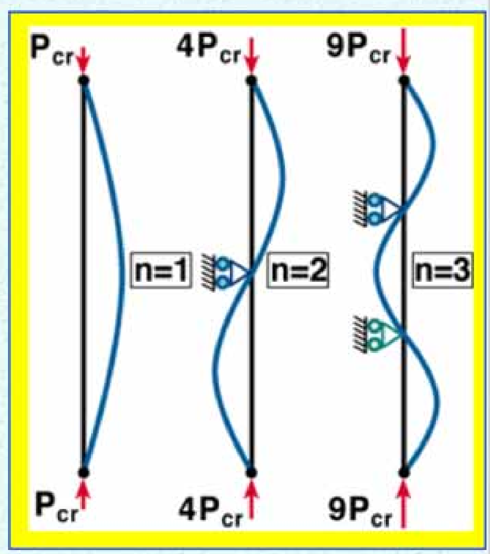


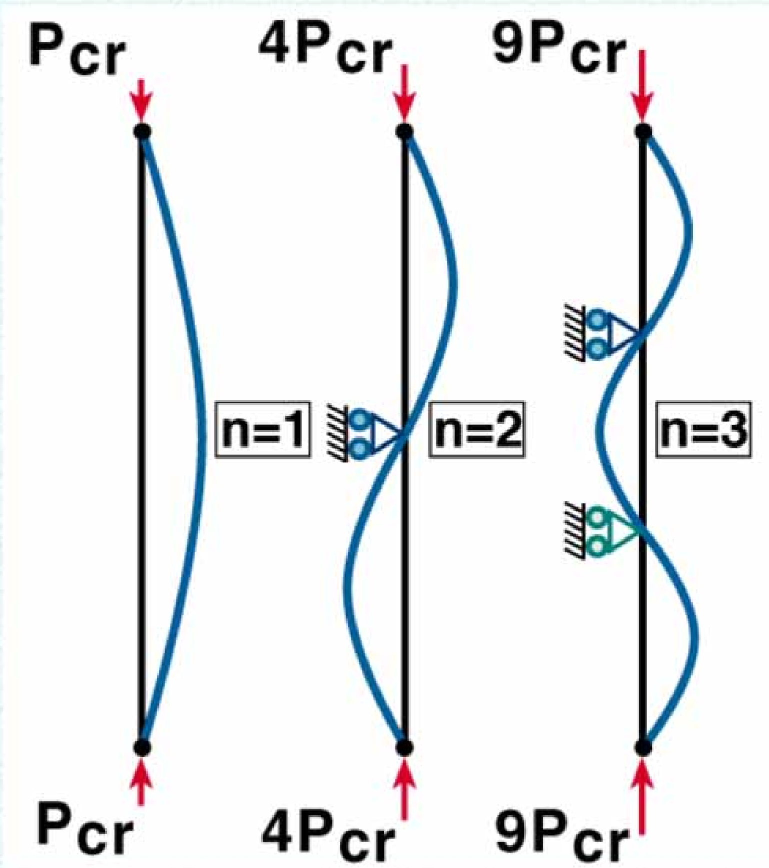
Stability of Columns (or Struts)

Higher buckling loads

$$\boxed{\begin{array}{l} n = 2 \longrightarrow P = 4\,\dfrac{\pi^2 EI}{L^2} \\ n = 3 \longrightarrow P = 9\,\dfrac{\pi^2 EI}{L^2} \end{array}}$$

$$v = A \sin\frac{n\pi z}{L}$$

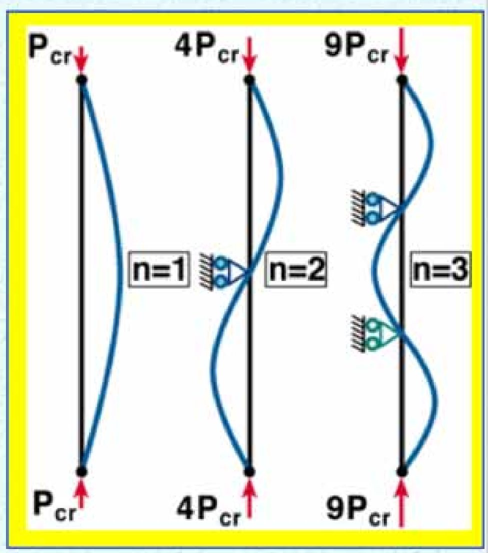




Stability of Columns (or Struts)

- $v = A \sin \frac{n\pi z}{L}$

Note that the homogeneous differential and the homogeneous boundary conditions characterize an eigenvalue problem. The values of $\sqrt{\frac{P}{EI}}$ for nontrivial solution $(v \neq 0)$ are called eigenvalues, and the associated v's are called eigenfunctions.



Stability of Columns (or Struts)

Energy Method

Axial shortening (displacement) of the column is given by

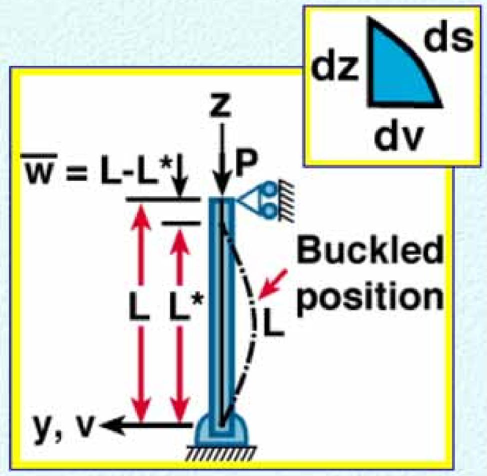
$$\overline{w} - L - L^*$$

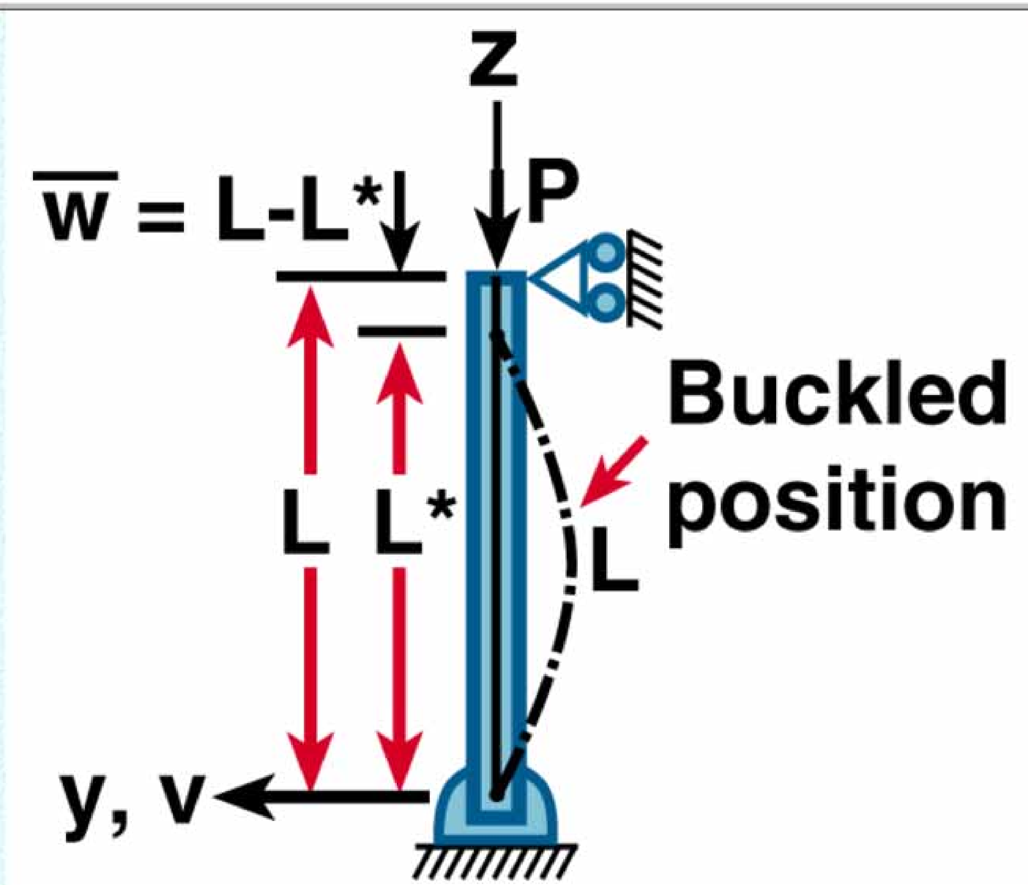
where

$$L = \int_0^L ds, \qquad L^* = \int_0^L dz$$

but,

$$(ds)^2 = (dz)^2 + (dv)^2$$





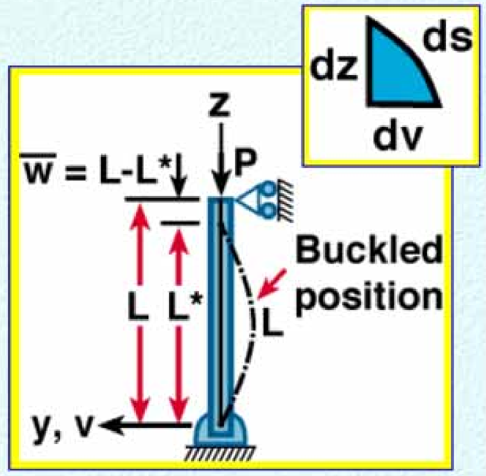
Stability of Columns (or Struts)

but,

$$(ds)^2 = (dz)^2 + (dv)^2$$

$$ds = \sqrt{1 + \left(\frac{dv}{dz}\right)^2}\, dz \simeq \left[1 + \frac{1}{2}\left(\frac{dv}{dz}\right)^2\right] dz$$

$$\bar{w} = \frac{1}{2}\int_0^L \left(\frac{dv}{dz}\right)^2 dz$$



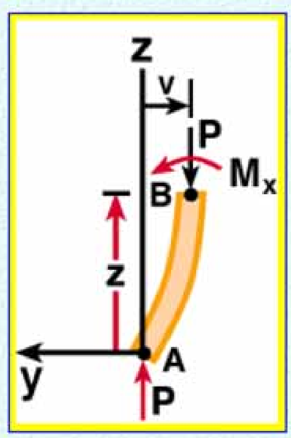
Stability of Columns (or Struts)

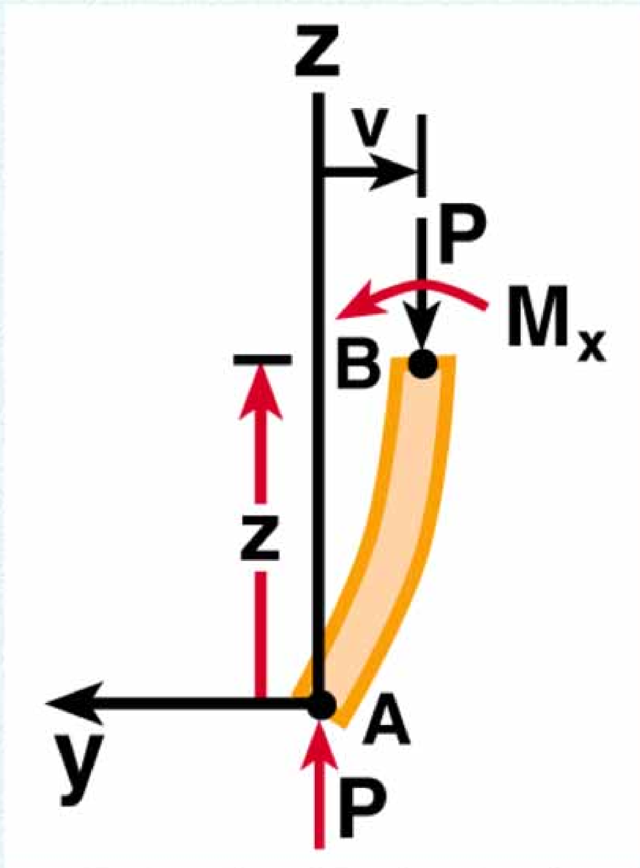
Δw= work done by the external force P

$$= \frac{P}{2}\int_0^L \left(\frac{dv}{dz}\right)^2 dz$$

Δu= strain energy of the beam

$$= \int_0^L \frac{1}{2} M \kappa \, dz$$



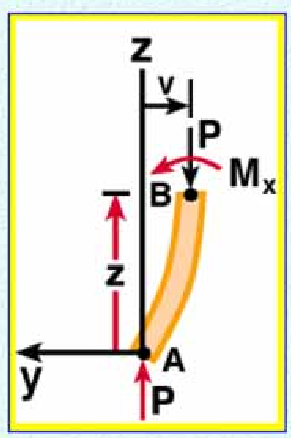


Stability of Columns (or Struts)

where

$$\kappa \cong -\frac{d^2v}{dz^2}$$

$$M = -EI\,\frac{d^2v}{dz^2}$$



$$\Delta U = \frac{1}{2}\int_0^L EI\left(\frac{d^2v}{dz^2}\right)^2 dz$$

Stability of Columns (or Struts)

Assuming $v = A \sin \frac{n\pi z}{L}$

which satisfies the boundary conditions of v = 0 at z = 0 and z = L

$$\Delta W = \frac{\pi^2 P}{4L} n^2 A^2$$

$$\Delta U = \frac{\pi^4 EI}{4L^3} n^4 A^2$$

Stability of Columns (or Struts)

which satisfies the boundary conditions of v = 0 at z = 0 and z = L

$$\Delta W = \frac{\pi^2 P}{4L} n^2 A^2$$

$$\Delta U = \frac{\pi^4 EI}{4L^3} n^4 A^2$$

Bifurcation buckling load corresponds to: $\Delta U = \Delta W$

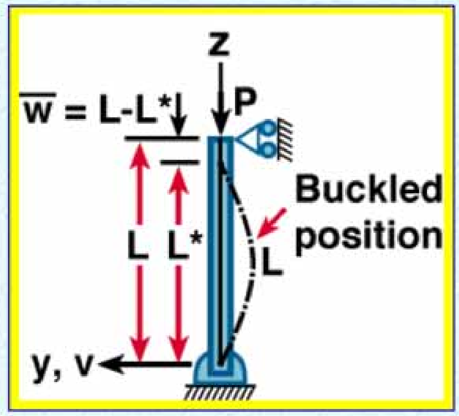
or $$P_{cr} = \frac{n^2 \pi^2 EI}{L^2}$$

Critical Stresses in Columns

If at buckling the material of the column is stressed within the elastic range, then

$$\sigma_{cr.} = \frac{P_{cr}}{A}$$

$$= \frac{\pi^2 EI}{L^2} / A$$



But $I = A\,r^2$ where r = radius of gyration of the cross section.

Then $$\sigma_{cr} = \frac{\pi^2 E}{(L/r)^2}$$

where L/r is called the slenderness ratio.

Eccentrically Loaded Columns

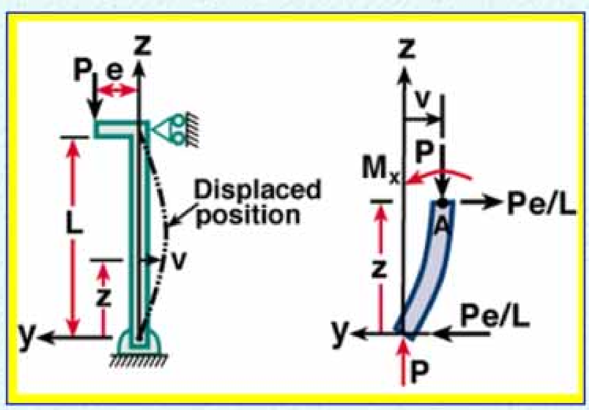
If the column is loaded eccentrically - the load is displaced at distance e from the centerline. Then

$$M_x = P\left(v + \frac{e}{L} z\right)$$

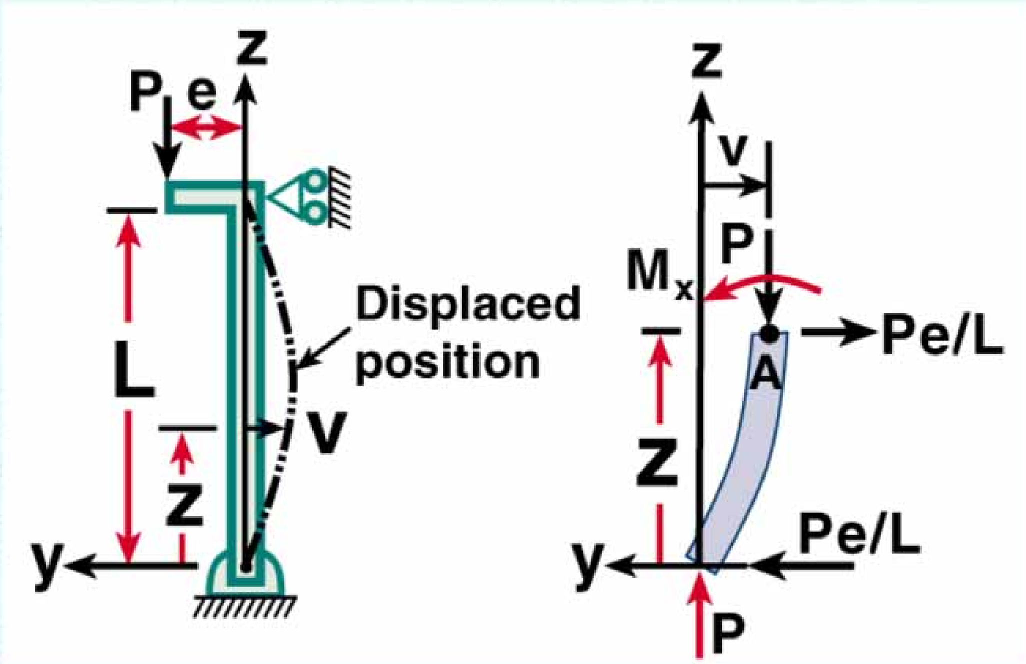
$$= -EI \frac{d^2 v}{dz^2}$$

or

$$\frac{d^2 v}{dz^2} + \frac{P}{EI} v = -\frac{P}{EI} ez$$



which is a nonhomogeneous ordinary differential equation.



Eccentrically Loaded Columns

Boundary Conditions

At z = 0, L , v = 0

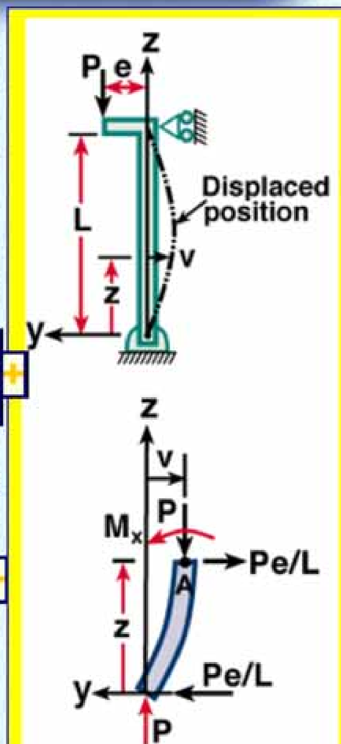
homogeneous boundary conditions.

General solution of the differential equation is:

$$v = A \sin\sqrt{\frac{P}{EI}}\,z + B \cos\sqrt{\frac{P}{EI}}\,z - \frac{e}{L}z$$

Applying the boundary conditions, then

$$v = e\left(\sin\sqrt{\frac{P}{EI}}\,z \Big/ \sin\sqrt{\frac{P}{EI}}\,L - \frac{z}{L}\right)$$

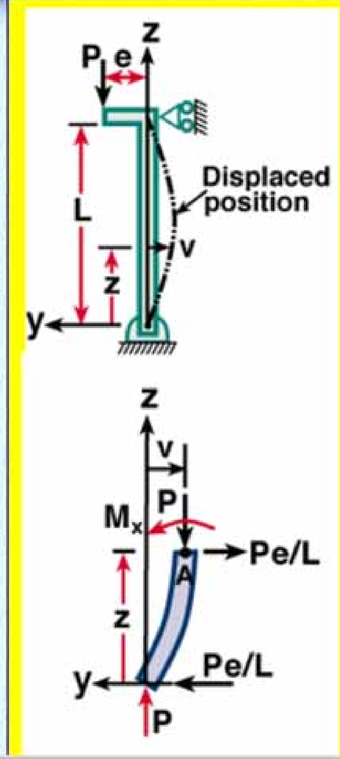


Eccentrically Loaded Columns

For certain values of P, the denominator

$$\sin\sqrt{\frac{P}{EI}}L$$

becomes zero, and the deflection becomes infinitely large. The corresponding values of P are the critical buckling loads.



Eccentrically Loaded Columns

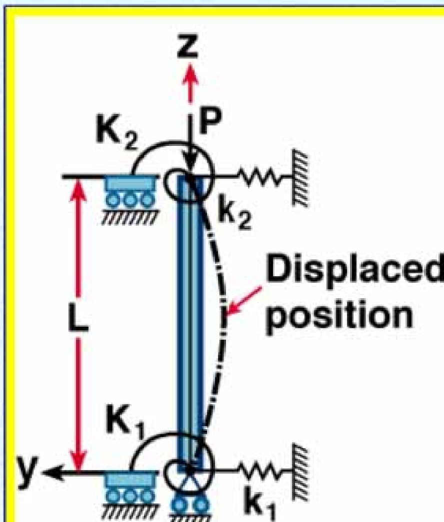
<u>Axially Loaded Slender Columns with General End Restraints</u>

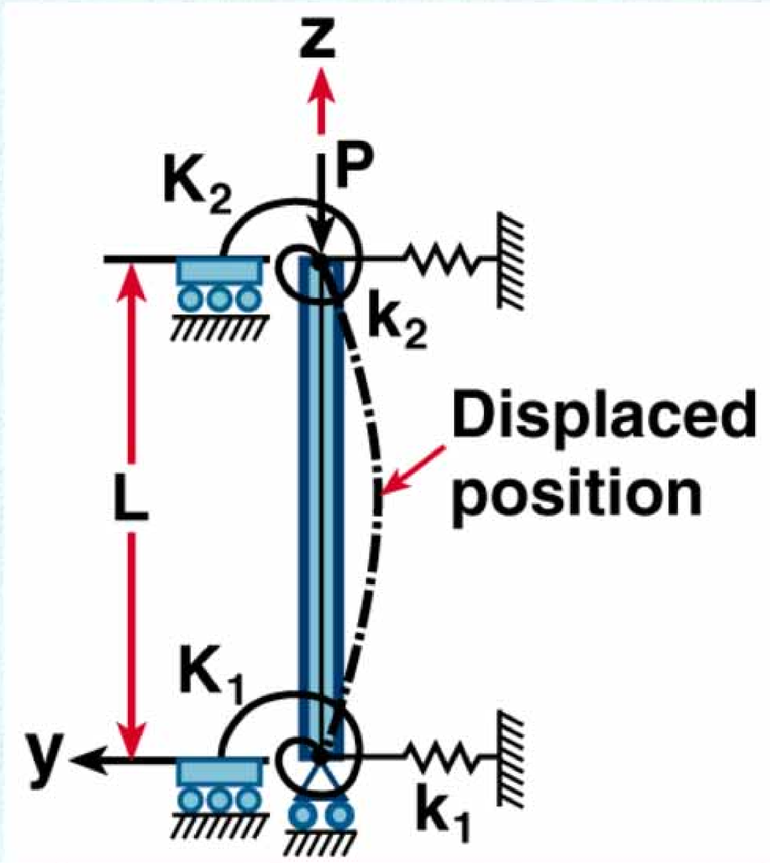
If the governing homogeneous differential equation for an axially loaded column, with constant EI

$$\frac{d^2v}{dz^2} + \frac{P}{EI}v = 0$$

is differentiated twice with respect to z,, the following fourth-order differential equation is obtained:

$$\frac{d^4v}{dz^4} + \frac{P}{EI}\frac{d^2v}{dz^2} = 0$$





Eccentrically Loaded Columns

The general solution of the differential equation is:

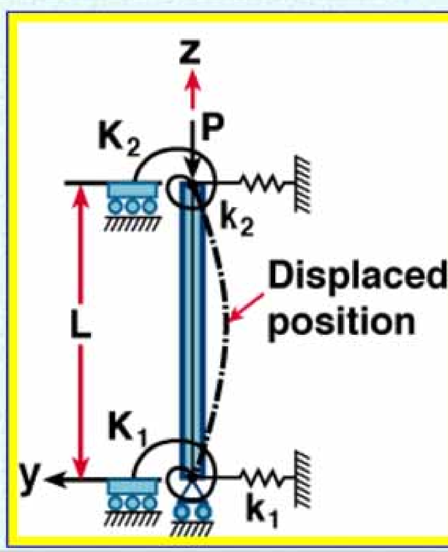
$$v = A \sin\sqrt{\frac{P}{EI}}\,z + B \cos\sqrt{\frac{P}{EI}}\,z + Cz + D$$

where A, B, C, and D are constants, to be determined from four boundary conditions - two at each end of the column.

The boundary conditions specify:

displacement **v**

slope (or rotation) $\frac{dv}{dz}$



Eccentrically Loaded Columns

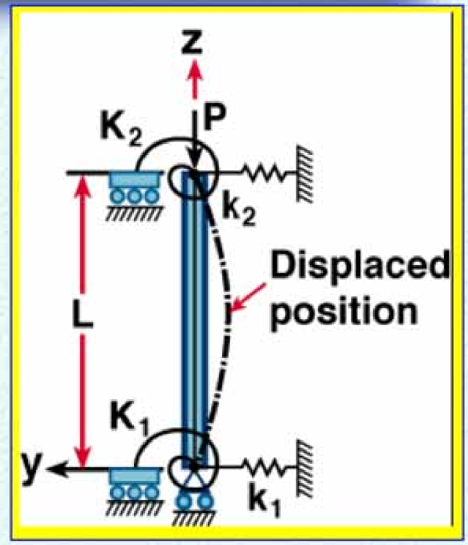
The boundary conditions specify:

- displacement **v**
- slope (or rotation) $\frac{dv}{dz}$
- bending moment

$$M_x = -EI\frac{d^2v}{dz^2}$$

shearing force

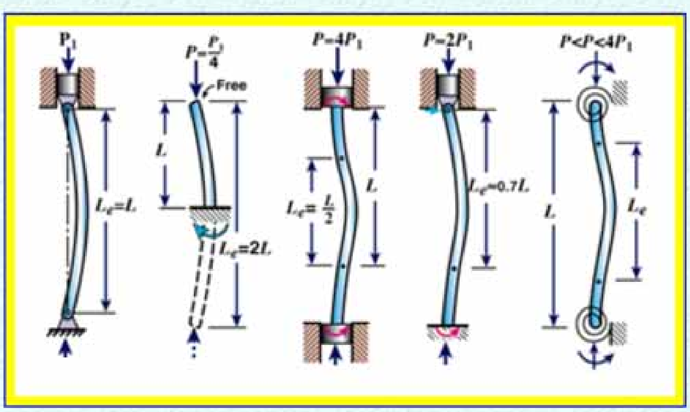
$$V_y = -EI\frac{d^3v}{dz^3}$$

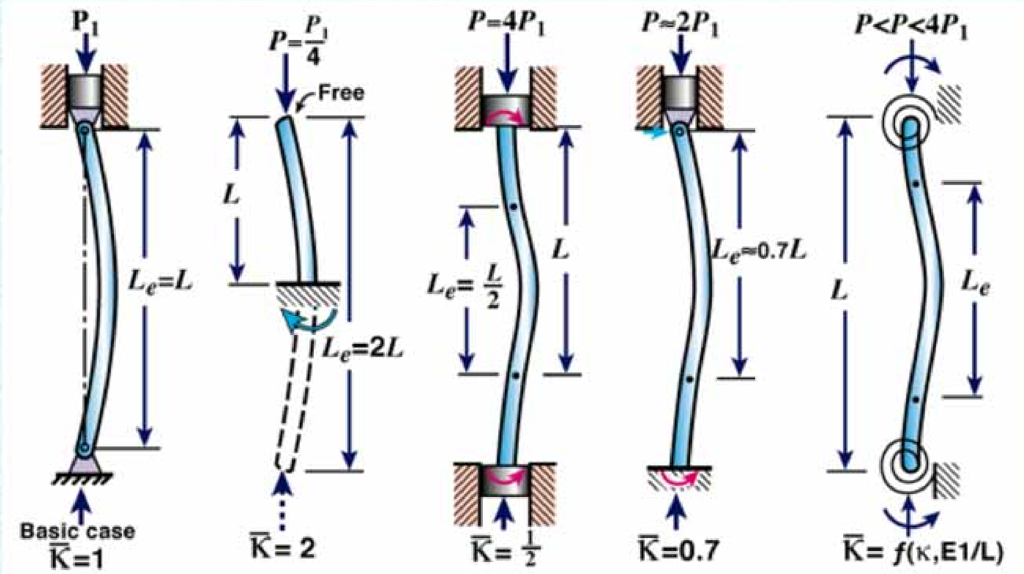


Effect of End Restraints on Buckling Loads

Basic Case - simply supported coumn (Euler buckling load)

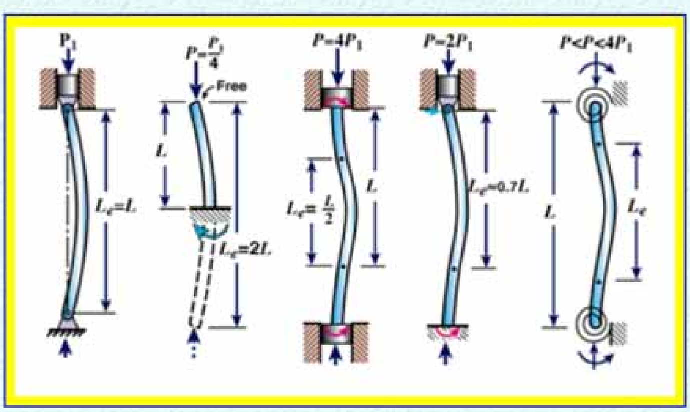
$$\boxed{P_{cr} = \frac{\pi^2 EI}{L^2}}$$





Effect of End Restraints on Buckling Loads

End constraints can be accounted for by finding the effective length L_e (length of a simply supported column) that would have the same critical load as that of the original column.

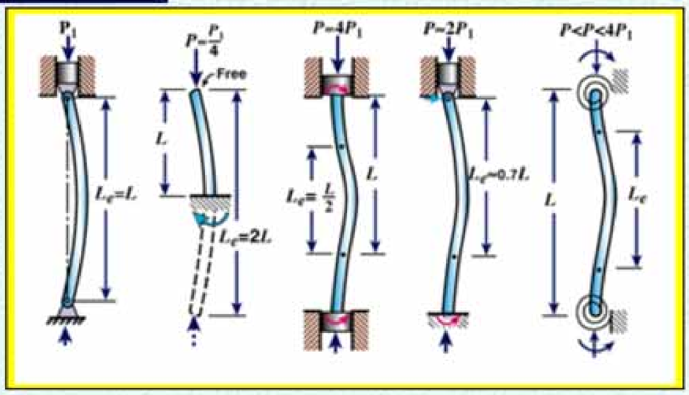


Effect of End Restraints on Buckling Loads

$$P_{cr} = \frac{\pi^2 EI}{(L_e)^2}$$

$$\sigma_{cr} = \frac{\pi^2 E}{(L_e/r)^2}$$

where $\boxed{L_e = \bar{\kappa} L}$



Basic case $\bar{K}=1$ | $\bar{K}=2$ | $\bar{K}=\frac{1}{2}$ | $\bar{K}=0.7$ | $\bar{K}=f(\kappa, E1/L)$

Inelastic Column Theory

The critical σ

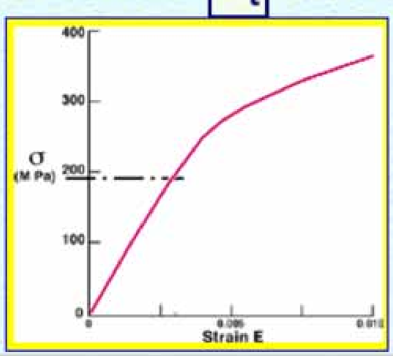
$$\sigma_{cr} = \frac{\pi^2 E}{(L_e/r)^2}$$

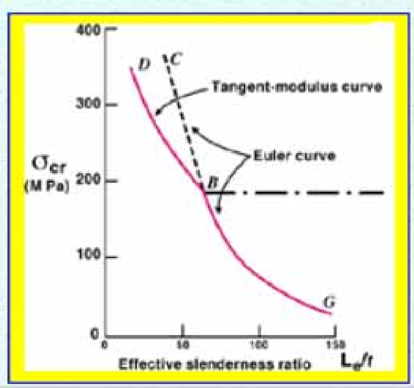
assumes that buckling occurs before yielding (columns with sufficiently large L_e/r)

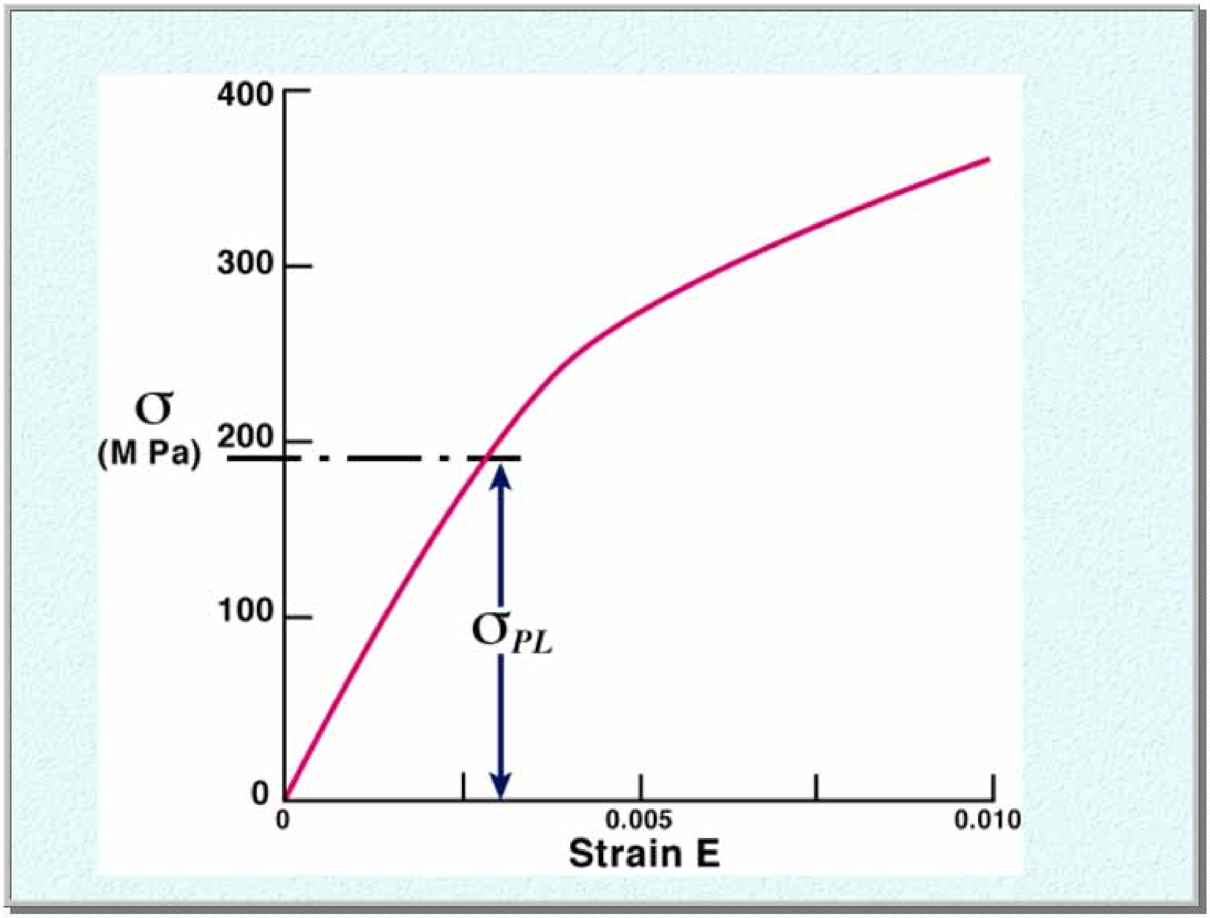
For small values of $\boxed{L_e/r}$ Engesser suggested replacing the Young's modulus $\boxed{E}$ by the tangent modulus $\boxed{E_t}$ where

$$E_t = \frac{d\sigma}{d\varepsilon}$$

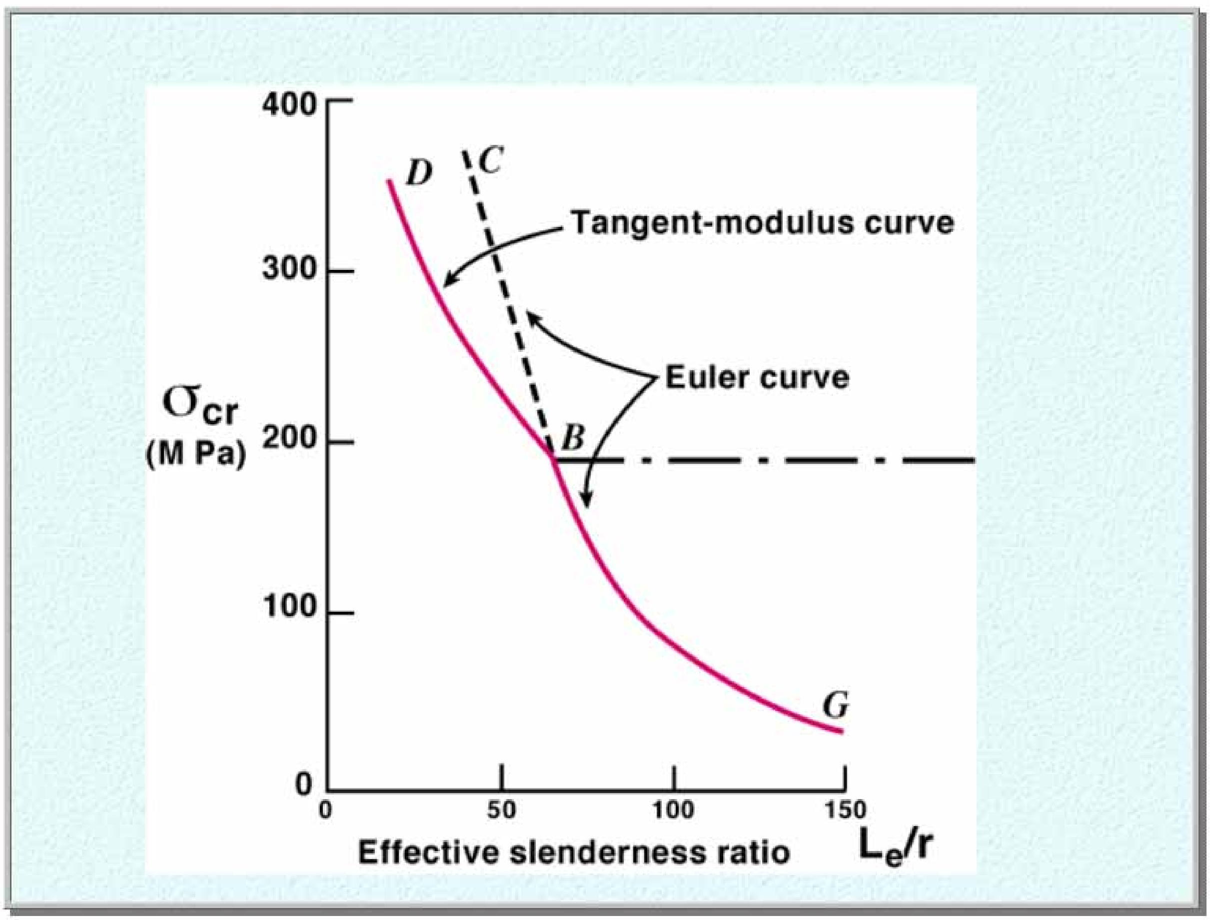
$$\sigma_{cr} = \frac{\pi^2 E_t}{(L_e/r)^2}$$







400
300
σ
(M Pa)
200
100
0
σ_PL
0
0.005
0.010
Strain E

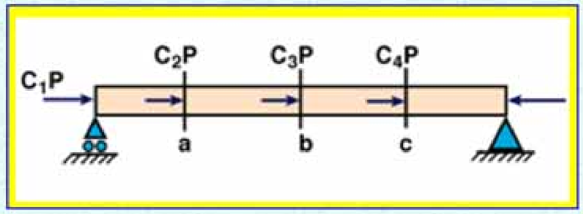


400
D
C
Tangent-modulus curve
300
σcr
(M Pa)
Euler curve
200
B
100
G
0
0
50
100
150
Effective slenderness ratio
Le/r

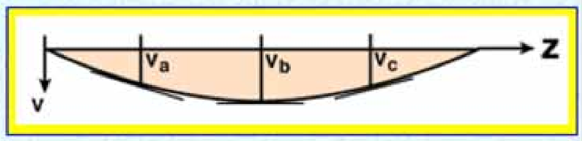
Inelastic Column Theory

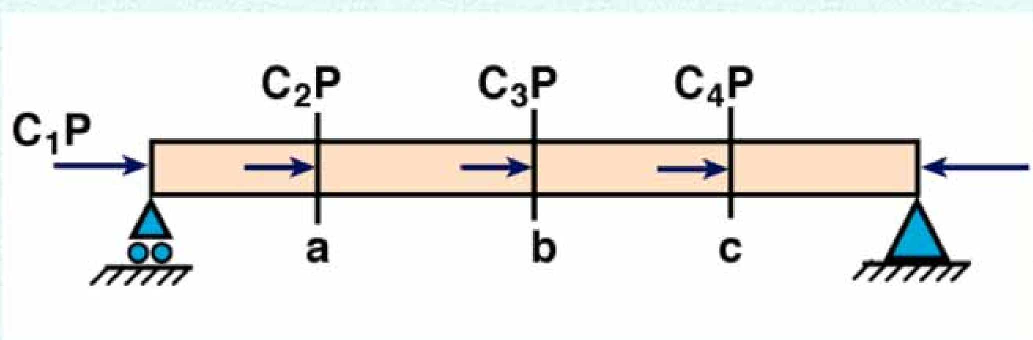
Columns Subjected to More Than One Concentrated Load

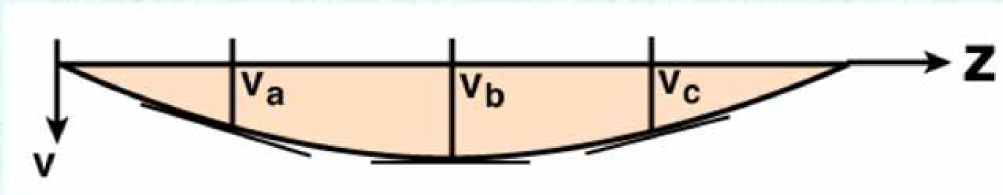
1. Write the differential equation for each beam segment and find the general solution.



2. Apply both the boundary conditions at the supports and the continuity conditions at the points of application of concentrated loads.





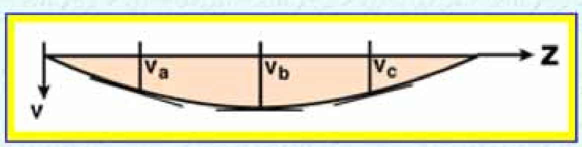


Inelastic Column Theory

2. Apply both the boundary conditions at the supports and the continuity conditions at the points of application of concentrated loads.

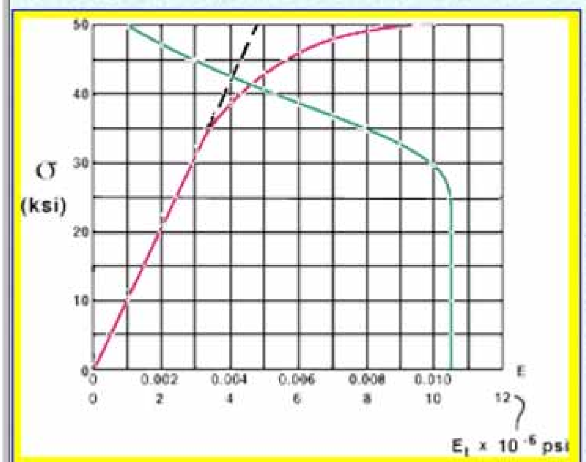
$$\boxed{v\big|_L = v\big|_r}$$

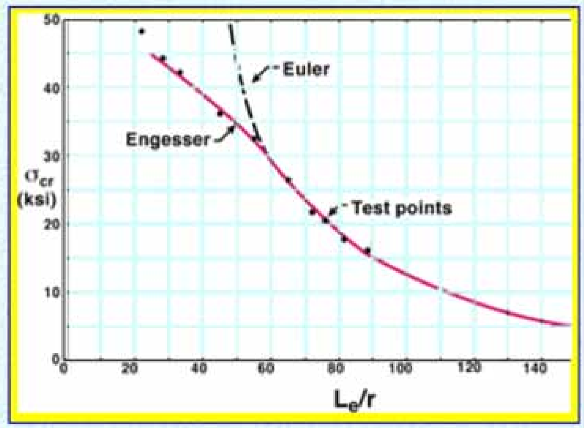
$$\boxed{\left.\frac{dv}{dz}\right|_L = \left.\frac{dv}{dz}\right|_r}$$

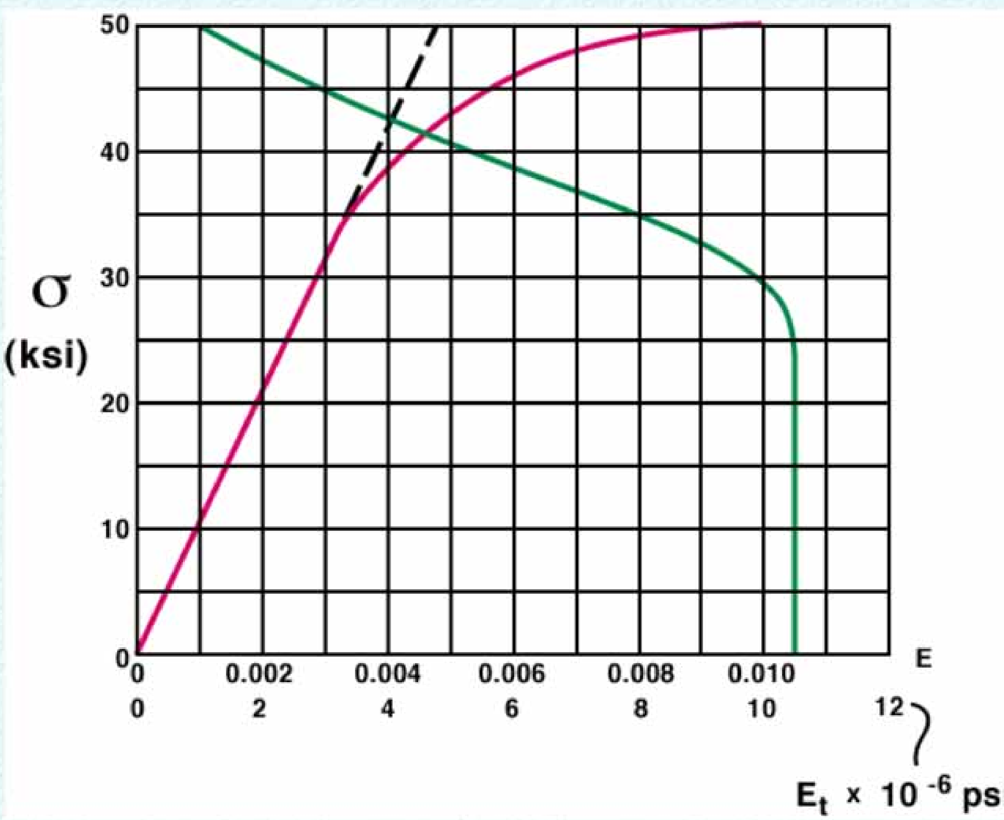


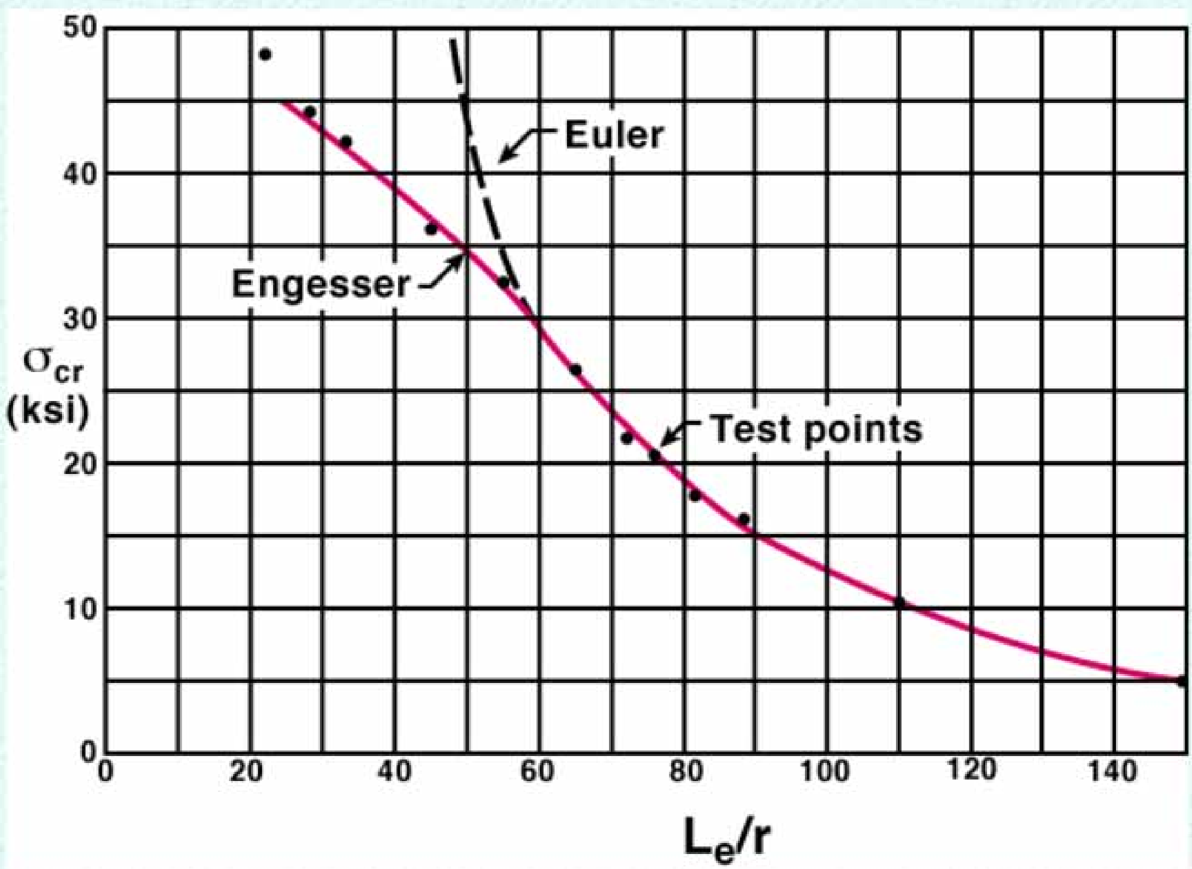
Inelastic Column Theory

3. The resulting homogeneous equations in step 2 define the eigenvalue problem, from which the buckling load can be obtained.









50
40
30
20
10
0
σ_{cr}
(ksi)
Euler
Engesser
Test points
0
20
40
60
80
100
120
140
L_e/r